Practice Test 4/13/15 for 313 sections 04/05

Part I: Multiple Choice Questions: *Mark the correct answers for each question*

1. Find a basis of $W^{\perp}$ where W has basis $B := \{[1,1,1,1]^T, [1,2,0,0]^T\}$

2. A certain experiment produces the data points $(4,8), (9,15)$, and $(16,24)$. These data points lie on a curve of the form

$$y = \beta_0 x + \beta_1 \sqrt{x}.$$

The parameters β_0 and β_1 can be found by solving which of the following matrix equations:

a) $\begin{bmatrix} 4 & 9 & 16 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 15 \\ 24 \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$

b) $\begin{bmatrix} 4 & 2 \\ 9 & 3 \\ 16 & 4 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 24 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 4 \\ 1 & 9 \\ 1 & 16 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 24 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 8 \\ 1 & 15 \\ 1 & 24 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix}$

3. Which of the following is FALSE?

   a) Symmetric matrices are always diagonalizable.

   b) Symmetric matrices are always invertible.

   c) Eigenvectors of a symmetric matrix which come from different eigenspaces must be orthogonal.

   d) Eigenvectors of a symmetric matrix which come from different eigenspaces must be linearly independent.

4. Select the matrix corresponding to the quadratic form

$$Q(x_1, x_2, x_3, x_4) = -2x_1^2 + 5x_2^2 - 7x_3^2 - 4x_4^2 - 6x_1x_2 + 4x_3x_1 - 10x_1x_4$$

a) $\begin{bmatrix} -2 & 5 & -7 & -4 \\ 2 & 3 & 4 & -10 \\ 5 & -7 & -4 & -6 \\ 2 & -3 & -5 & 0 \end{bmatrix}$

b) $\begin{bmatrix} -2 & -3 & 2 & -5 \\ -3 & 5 & 0 & 0 \\ 2 & 0 & -7 & 0 \\ -5 & 0 & 0 & -4 \end{bmatrix}$

c) $\begin{bmatrix} -2 & -6 & 4 & -10 \\ -6 & 5 & 0 & 0 \\ 4 & 0 & -7 & 0 \\ -10 & 0 & 0 & -4 \end{bmatrix}$

d) $\begin{bmatrix} -2 & -6 & 4 & -10 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

5. Which of the sets is orthogonal under the given inner product on $C[0, \pi]$?

$$\langle f, g \rangle = \int_0^{\pi} f(x)g(x)dx$$

a) $\{1, \sin x\}$

b) $\{1, \cos x\}$

c) $\{1, -1\}$

d) $\{\sin x, \cos x\}$

6. Let $Q(x) = x^T Ax$ be a quadratic form, with A a symmetric matrix. Which statements are always true?

a) Q is positive definite

b) Introducing a new variable by setting $x = Py$ allows eliminating mixed terms.

c) A is orthogonally diagonalizable.

d) If $A = B^T B$ then A is positive semidefinite.

e) If $A = B^T B$ then A is positive definite.

7. Let A and B be $n \times n$ matrices. Which of the following is FALSE?

a) $\det(AB) = \det A \det B$

b) $\det(A^T) = \det A$

c) $\det(A^{-1}) = \det A$

d) $\det(kA) = k^n \det A$

8. What is the minimum value of $\mathbf{x}^T A\mathbf{x}$ subject to $\mathbf{x}^T\mathbf{x}$=1, if $A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$

a) 0

b) -3

c) 3

d) -7

e) 7

Part II: *Fill in the blank with the* ***best*** *possible answer. (x points each.)*

9. The Cauchy-Schwarz Inequality states that for all $\mathbf{u}, \mathbf{v}$ in a vector space V, ______________ $\leq$ ______________.

10. Let A, B, and C be invertible $n \times n$ matrices. Then the inverse of $AB^{-1}C^TA^{-1}B$ is equal to ___________________.

Part III: *Justify your answer and show all work for full credit.*

11. Find the equation of the least-squares line that best fits the data points $(-1,-2), (0,1), (1,1), (2,1)$, and $(3,4)$.

12. The matrix $A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$ has eigenvalues -2 and 1. Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

13. Let W be the subspace of $\mathbb{P}_2$ spanned by $\{t, t^2\}$. Find the orthogonal projection of 1 onto W using the inner product

$$\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

14. Prove that if λ is an eigenvalue of the matrix A, then λ^2 is an eigenvalue of the matrix A^2.

15. Compute the eigenvalues and eigenvectors corresponding the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

16. Prove that similar matrices have the same determinant.

17. Prove the Pythagorean Theorem: If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $\mathbb{R}^n$, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}|^2 + \|\mathbf{v}\|^2$$

18. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. Note that $\mathcal{B}$ is a basis for $\mathbb{R}^2$. (You do not need to prove this.

    (a) Find the change-of-coordinates matrix $P_{\mathcal{B}}$ from the basis $\mathcal{B}$ to the standard basis.

    (b) What is the equation relating $\mathbf{x}$, $[\mathbf{x}]_{\mathcal{B}}$, and $P_{\mathcal{B}}$?

    (c) Find the vector $\mathbf{x}$ if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

    (d) Use an inverse matrix to find $[\mathbf{y}]_{\mathcal{B}}$ if $y = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

19. For given A and $\mathbf{b}$ compute all solutions $\hat{\mathbf{x}}$ of the least squares problem and $\text{dist}(A\hat{\mathbf{x}}, \mathbf{b})$.

(a) $A := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) $A := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

(c) $A := \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

20. Find the inverse of the matrix

$$A = \begin{bmatrix} -3 & 4 & 0 \\ -2 & 3 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

using any method of your choosing.

21. Let A, C, and D be $n \times n$ matrices, with $CA = I$ and $AD = I$. Prove that $C = D$.

22. Let A be the transpose of $B := \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$. Find U, V, D and Σ for an SVD of A and prove its correctness.

END OF EXAM