# 11.6 Power Series and Laurent Series

March 23, 2020

Throughout we assume that $(X, \|\cdot\|)$ is a complex Banach space.

For a function $f : U \to X$ holomorphic on an open $U$ in $\mathbb{C}$, we can form for each $z_0 \in U$ the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

because $f$ is infinitely holomorphic on $U$ by Corollary 11.4.8.

Using the geometric series, and Cauchy's Integral and Differentiation formulas the following result shows that the Taylor series of $f$ at $z_0$ converges to $f$ on some open ball $B(z_0, r) \subset U$ for $r > 0$.

Theorem 11.6.1. For an open $U$ in $\mathbb{C}$, if

- $f : U \to X$ is holomorphic,

then

- for each $z_0 \in U$ there exists a largest $r \in (0, \infty]$ such that $B(z_0, r) \subset U$, and
- the Taylor series for $f$ at $z_0$ converges uniformly to $f$ on compact subset of $B(z_0, r)$.

[Draw the picture of $U$ and $B(z_0, r)$]

Remark 11.6.2. We now have shown that a function $f : U \to X$ is holomorphic on $U$ if and only if $f$ is analytic on $U$. Because of this equivalence, we often use holomorphic and analytic interchangeably.

Proposition 11.6.3. For $\{a_k\}_{k=0}^{\infty} \subset X$, a convergent power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

about $z_0 \in \mathbb{C}$ is unique and equal to its Taylor series.

Corollary 11.6.4. For an open, path-connected subset $U$ of $\mathbb{C}$, and a holomorphic $f : U \to X$, if there exists $z_0 \in U$ such that $f^{(n)}(z_0) = 0$ for all $n = 0, 1, 2, 3, \ldots$, then $f(z) = 0$ for all $z \in U$.

# What questions do you have?

First Online Reading Quiz Question.

(1) What are you reading about the zeros of analytic functions?

(2) What would happen if an analytic function $f$ had infinitely many zeros on a compact set inside is open domain $U$?

A property of a not identically equal to zero holomorphic function $f : U \to X$ is that its zeros, if any, must be "isolated."

This is to say, if $f(z_0) = 0$, then there exists $\epsilon > 0$ such that $f(z) \neq 0$ for all $z \in B(z_0, \epsilon) \setminus \{z_0\}$.

We will use the "order of a zero" (defined next) to obtain this isolation of zeros.

Definition 11.6.5. For an open $U$, we say that a holomorphic $f : U \to X$ has a zero of order $n \in \mathbb{N}$ at $z_0 \in U$ if the Taylor series of $f$ about $z_0$ has the form

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k$$

for $a_n \neq 0$, i.e., $f^{(j)}(z_0) = 0$ for all $j = 0, 1, \ldots, n-1$ and $f^{(n)}(z_0) \neq 0$.

Proposition 11.6.6. For an open $U$ and $f : U \to X$ holomorphic, if $z_0 \in U$ is a zero of order $n$ for $f$, then there exists a holomorphic function $g : U \to X$ such that

$$f(z) = (z - z_0)^n g(z)$$

and

$$g(z_0) \neq 0,$$

and there exists $\epsilon > 0$ such that $B(z_0, \epsilon) \subset U$ and $f(z) \neq 0$ for all $z \in B(z_0, \epsilon) \setminus \{z_0\}$.

Remark. This says the you can literally factor out of $f$ the term $(z - z_0)^n$ that completely accounts for the zero of $f$ at $z_0$, and the order is the power $n$ of $z - z_0$ in this factor.

Example. The entire function $f(z) = \sin(z)$ has an isolated zero $z_0 = 0$.

From the Taylor series we have

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = (z-0) \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!} = (z - z_0) g(z),$$

where $g(0) \neq 0$, so that $f(z)$ has a zero of order 1 at $z_0 = 0$.

Corollary 11.6.7 (Local Isolation of Zeros). For an open, path-connected $U$ in $\mathbb{C}$, and a holomorphic $f : U \to X$, if

- there is a sequence $(z_k)_{k=1}^{\infty}$ of distinct points in $U$ where $z_k \to w \in U$ and $f(z_k) = 0$ for all $k \in \mathbb{N}$,

then

- $f(z) = 0$ for all $z \in U$.

# What questions do you have?

## A journey to Laurent Series

For a holomorphic $f : U \to X$ such that $f(z_0) \neq 0$ for $z_0 \in U$, the function

$$g(z) = \frac{f(z)}{z - z_0}$$

is not complex differentiable at $z_0$ and so there is no Taylor series for $g$ about $z_0$.

We know by Cauchy's Integral formula that for any simple closed contour $\gamma$ in $U$ enclosing $z_0$ there holds

$$\frac{1}{2\pi i} \oint_\gamma g(z)\ dz = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\ dz = f(z_0).$$

On the other hand....

Using the Taylor's series for $f$ about $z_0$, i.e.,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k,$$

we can express the function $g$ as

$$g(z) = \frac{f(z)}{z - z_0} = \frac{1}{z - z_0} \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$
$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^{k-1}$$

which makes sense on a punctured ball $B(z_0, r) \setminus \{z_0\}$ for some $r > 0$.

Since $z_0 \notin \gamma$ and integration and uniform convergence commute, we can use the series expression for $g$ to compute

$$\oint_\gamma g(z)\, dz = \oint_\gamma \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-1} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} \oint_\gamma (z-z_0)^{k-1}.$$

By Lemma 11.3.5, the contour integrals

$$\oint_\gamma (z-z_0)^{k-1}\, dz = 0 \text{ when } k \geq 1, \text{ and}$$

$$\oint_\gamma (z-z_0)^{k-1}\, dz = 2\pi i \text{ when } k = 0.$$

Thus we obtain

$$\frac{1}{2\pi i}\oint_\gamma g(z)\, dz = \frac{2\pi i}{2\pi i} f(z_0) = f(z_0)$$

in agreement with Cauchy's Integral Formula.

# What questions do you have?

Definition. For coefficients $a_k \in X$ for $k \in \mathbb{Z}$, a Laurent series is a series of the form

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k.$$

To talk about convergence of Laurent series we will use the open annulus $A$ centered at $z_0$ with inner radius $r$ and outer radius $R$ defined by

$$A = \{z \in \mathbb{C} : r < |z - z_0| < R\}$$

where $0 \leq r < R \leq \infty$.

[Draw picture of open annulus]

Theorem 11.6.8 (Laurent Expansion). For the open annulus $A$ centered at $z_0$ with inner radius $r$ and outer radius $R$, if $f : A \to X$ is holomorphic, then $f$ has a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

where each of the power series in the decomposition

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} a_{-k} \left( \frac{1}{z - z_0} \right)^k$$

converge uniformly and absolutely on every compact subannulus

$$D_{\rho,\varrho} = \{ z \in \mathbb{C} : \rho \le |z - z_0| \le \varrho \}$$

of $A$, i.e., for all $r < \rho < \varrho < R$. The coefficients $a_k$ in the Laurent series for $f$ are given explicitly by

$$a_k = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^{k+1}} \, dw$$

for any circle $\gamma$ of radius strictly between $r$ and $R$.

## What questions do you have?

Second Reading Quiz Question: Computing a Laurent series is just as easy as computing a Taylor series.

False

Remark 11.6.10. Computing the Laurent series is usually quite difficult.

We will learn two ways to find Laurent series (illustrated by way of upcoming examples).

As we will see, the only coefficient we really need in the Laurent series of $f$ holomorphic on the annulus $A = B(z_0, \epsilon) \setminus \{z_0\}$ is that of the term $(z - z_0)^{-1}$ when computing any contour integral

$$\oint_\gamma f(z)\, dz$$

for a simple closed contour $\gamma$ in $A$ that encloses $z_0$.

Example (in lieu of 11.6.11). The Laurent series of

$$f(z) = \frac{\sin(z)}{z^4}$$

on the open annulus

$$A = \{z \in \mathbb{C} : 0 < |z| < \infty\} = \mathbb{C} \setminus \{0\}$$

is obtained by dividing the power series for $\sin(z)$ by $z^4$, i.e.,

$$\begin{aligned}
\frac{\sin(z)}{z^4} &= \frac{1}{z^4} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k-3}}{(2k+1)!} \\
&= \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{5!} - \cdots .
\end{aligned}$$

Since the Laurent series for this holomorphic function converges uniformly on $A$, we can compute the contour integral of $f$ over any circle centered at 0 with radius $\nu > 0$ by a "direct" calculation after commuting the sum and the integral:

$$\begin{aligned}\oint_\gamma f(z)\ dz &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \oint_\gamma (z - z_0)^{2k-3}\ dz \\ &= -\frac{1}{6} \oint_\gamma (z - z_0)^{-1}\ dz \\ &= -\frac{\pi i}{3},\end{aligned}$$

where all the other contour integrals are zero by Lemma 11.3.5.

This gives

$$\frac{1}{2\pi i}\oint_{\gamma} f(z)\ dz = -\frac{1}{6}.$$

On the other hand, by Cauchy's Differentiation formula we arrive at the same answer:

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\gamma} f(z)\ dz &= \frac{1}{3!}\frac{3!}{2\pi i}\oint_{\gamma}\frac{\sin(z)}{(z-0)^4}\ dz \\ &= \frac{1}{6}\big(-\cos(0)\big) \\ &= -\frac{1}{6}\end{aligned}$$

because the third derivative of $\sin(z)$ is $-\cos(z)$.

# What questions do you have?

Example (in lieu of 11.6.12). Find the Laurent series for

$$f(z) = \frac{2}{(z-1)^2(z+1)}$$

about the point $z_0 = 1$, i.e., an open annulus centered at $z_0 = 1$. [We will determine the inner and outer radius in a moment.]

Applying the method of partial fractions to the function gives

$$f(z) = \frac{1}{(z-1)^2} - \frac{1/2}{z-1} + \frac{1/2}{z+1}.$$

We express the last term

$$\frac{1/2}{z+1}$$

as a power series in $(z-1)$ using the geometric series as follows:

$$
\begin{aligned}
\frac{1/2}{z+1} &= \frac{1/2}{2-(-z+1)} \\
&= \frac{1/4}{1-(-z+1)/2} \qquad \left[\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k, \quad r = \frac{-z+1}{2}\right] \\
&= \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{-z+1}{2}\right)^k \\
&= \frac{1}{4}\sum_{k=0}^{\infty}\left(-\frac{z-1}{2}\right)^k \\
&= \frac{1}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}(z-1)^k \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{k+2}}(z-1)^k.
\end{aligned}
$$

We obtain the Laurent series

$$f(z) = \frac{1}{(z-1)^2} - \frac{1/2}{z-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+2}} (z-1)^k.$$

The open annulus $A$ of convergence for this Laurent series is centered at $z_0 = 1$, with inner radius $r = 0$ and outer radius $R = 2$ which is determined by the condition for convergence of the geometric series

$$|(-z+1)/2| < 1.$$

[Draw picture of annulus $A$]

For a simple closed contour $\gamma$ in $A$ that encloses $z_0 = 1$ we use the Laurent series

$$f(z) = \frac{1}{(z-1)^2} - \frac{1/2}{z-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+2}} (z-1)^k$$

to compute

$$\frac{1}{2\pi i} \oint_{\gamma} f(z)\ dz = \frac{1}{2\pi i} \oint_{\gamma} \frac{-1/2}{z-1}\ dz = -\frac{2\pi i}{4\pi i} = -\frac{1}{2},$$

where we have used the interchange of integration and uniform convergence, and Lemma 11.3.5.

# What questions do you have?