11.7 The Residue Theorem

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As always $(X, \|\cdot\|)$ is a complex Banach space.

Here is an outline for today.

- Isolated Singularities
- Residues and Winding Numbers
- The Residue Theorem

First Reading Quiz Question:

- What are the three types of isolated singularities that a holomorphic function can have?
- How is the residue of a holomorphic function at an isolated singularity computed?

Definition 11.7.1. For a point $z_0 \in \mathbb{C}$, an $\epsilon > 0$, and the punctured open disk

$$U = \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\},$$

for $f : U \to X$ holomorphic, we say that z_0 is an isolated singularity of f if f is not assumed complex differentiable at z_0.

For an isolated singularity z_0 of f the principal part of the Laurent series

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

of f on $B(z_0, \epsilon) \setminus \{z_0\}$ is the series

$$\sum_{k=-\infty}^{-1} a_k (z - z_0)^k.$$

We use the principal part to classify isolated singularities.

Definition. An isolated singularity z_0 of f is called a removable singularity if the principal part of the Laurent series of f about z_0 is zero, i.e., $a_k = 0$ for all $k = -1, -2, -3, \ldots$.

If f has a removable singularity at z_0, then f extends to a holomorphic function on $B(z_0, \epsilon)$ by means of the power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ convergent on $B(z_0, \epsilon)$.

Example. The function

$$f(z) = \frac{\cos(z) - 1}{z^2} = \frac{1}{z^2} \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k-2}}{(2k)!}$$

$$= -\frac{1}{2} + \frac{z^2}{4!} + \cdots$$

has a removable singular at the isolated singularity $z_0 = 0$ of f.

Definition. An isolated singular z_0 of f is called a pole of order $N \in \mathbb{N}$ if the principal part of the Laurent series of f about z_0 has the form

$$f(z) = \sum_{k=-N}^{-1} a_k (z - z_0)^k,$$

i.e., $a_k = 0$ for all $k < -N$ in the Laurent series for f about z_0.

A pole of order 1 is called a simple pole.

Example. The function

$$f(z) = \frac{\sin(z)}{z^4} = \frac{1}{z^4} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{5!} + \cdots$$

has a pole of order $N = 3$ at the isolated singularity $z_0 = 0$ of f.

Definition. An isolated singularity z_0 of f is called an essential singularity if the principal part of the Laurent series for f about z_0 has infinitely many nonzero terms, i.e., $a_k \neq 0$ for infinitely many $-k \in \mathbb{N}$.

Example. The function

$$f(z) = \sin\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k (1/z)^{2k+1}}{(2k+1)!}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{-2k-1}}{(2k+1)!}$$
$$= \frac{1}{z} - \frac{1}{6z^3} + \frac{1}{5!z^5} - \cdots$$

has an essential singularity at the isolated singularity $z_0 = 0$ of f.

What questions do you have?

Definition 11.7.4. For an open set U in $\mathbb{C}$ and finitely many distinct points $z_1, \dots, z_n$ in U, a function

$$f : U \setminus \{z_1, \dots, z_n\} \to X$$

is called meromorphic if f is holomorphic on the open set $U \setminus \{z_1, \dots, z_n\}$ with f having poles at each z_i.

Example 11.7.5. For polynomials p and q with q not identically equal to 0, the rational function

$$p(z)/q(z),$$

in lowest terms (i.e., any common factors that p and q have have already been cancelled), is a meromorphic function on $\mathbb{C} \setminus \{z_1, \dots, z_k\}$ where $z_1, \dots, z_k$ are the distinct roots of q.

FYI: It is standard practice is always assume that a rational function is given in lowest terms, unless explicitly told otherwise.

Remark. We have already seen that the coefficient a_{-1} of the power $(z - z_0)^{-1}$ in the Laurent series of a function f holomorphic on a punctured disk $B(z_0, \epsilon) \setminus \{z_0\}$ is the quantity needed when computing contour integrals of f on simply closed curves with z_0 in its interior.

Because of the importance of this coefficient, we give it a name.

Definition 11.7.6. For a holomorphic $f : B(z_0, \epsilon) \setminus \{z_0\} \to X$ and simple close curve γ in $B(z_0, \epsilon) \setminus \{z_0\}$, the quantity

$$\frac{1}{2\pi i} \oint_{\gamma} f(z)\, dz$$

is called the residue of f at z_0 and is denoted by $\mathrm{Res}(f, z_0)$.

Proposition 11.7.7. If $f : B(z_0, \epsilon) \setminus \{z_0\} \to X$ is holomorphic, then $\mathrm{Res}(f, z_0)$ is the coefficient a_{-1} of the power $(z - z_0)^{-1}$ in the Laurent series of f about z_0.

Proposition 11.7.8. Suppose a holomorphic f has an isolated singularity at z_0.

(i) The isolated singularity at z_0 is removable if and only if $\lim_{z \to z_0} f(z)$ exists (as a complex number; the book inaccurately uses the term finite).

(ii) If for some nonnegative integer k the limit $\lim_{z \to z_0}(z - z_0)^k f(z)$ exists (as a complex number), then the isolated singularity z_0 of f is either a removable singularity or a pole of order equal to or less than k.

(iii) If the limit $\lim_{z \to z_0}(z - z_0) f(z)$ exists (as a complex number), then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0}(z - z_0) f(z).$$

The proof of this is HW (Exercise 11.29).

What questions do you have?

Journey to the Residue Theorem

Consider the contour integral

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{z - z_0}\ dz$$

for the closed contour $\gamma : [0, 2k\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + e^{i\theta}$ for a positive integer k.

Computing this contour integral gives

$$\frac{1}{2\pi i}\int_0^{2k\pi} \frac{1}{e^{i\theta}}(ie^{i\theta})\ d\theta = \frac{1}{2\pi}\int_0^{2k\pi} d\theta = \frac{2k\pi}{2\pi} = k.$$

The closed contour γ goes around z_0 in the counterclockwise direction k times while the residue of $1/(z - z_0)$ at z_0 is 1.

If this same curve γ is traversed in the clockwise direction, i.e., $\gamma(\theta) = z_0 + e^{-i\theta}$, then we would get $-k$ as the value of the contour integral.

Furthermore, if γ is closed contour that does not enclose z_0, then $1/(z - z_0)$ is holomorphic on a simply connected open set containing γ but not containing z_0, so that by the Cauchy-Goursat Theorem we have

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z - z_0} \, dz = 0.$$

[Draw the picture]

These observations motivate the notion of the winding number.

Definition 11.7.9. For a closed contour γ in $\mathbb{C}$ and z_0 a point of $\mathbb{C}$ not on γ, the winding number of γ with respect to z_0 is the quantity

$$I(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z - z_0}\, dz.$$

Lemma 11.7.12. For a simply connected open set U in $\mathbb{C}$, a closed contour γ in U, and a point $z_0 \in U$ not on γ, if

$$N(z) = \sum_{k=0}^{\infty} \frac{b_k}{(z - z_0)^k}$$

is uniformly convergent on compact subsets of $U \setminus \{z_0\}$, then there holds

$$\frac{1}{2\pi i} \oint_\gamma N(z)\, dz = \mathrm{Res}(N, z_0) I(\gamma, z_0).$$

The proof of this is HW (Exercise 11.30).

What questions do you have?

Theorem 11.7.13 (The Residue Theorem). For a simply connected U in $\mathbb{C}$ and finitely many points $z_1, \dots, z_n \in U$, if

$$f : U \setminus \{z_1, \dots, z_n\} \to X$$

is holomorphic and γ is a closed contour in $U \setminus \{z_1, \dots, z_n\}$, then

$$\frac{1}{2\pi i} \oint_\gamma f(z)\ dz = \sum_{j=1}^{n} \mathrm{Res}(f, z_j) I(\gamma, z_j).$$

Second Quiz Question: The Cauchy-Goursat Theorem and the Cauchy Integral Formula are special cases of the Residue Theorem.

True

The Residue Theorem has the Cauchy-Goursat Theorem as a special case.

When $f : U \to X$ is holomorphic, i.e., there are no points in U at which f is not complex differentiable, and γ in U is a simple closed curve, we select any $z_0 \in U \setminus \gamma$.

The residue of f at z_0 is 0 by Proposition 11.7.8 part (iii), i.e.,

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z) = 0;$$

hence, regardless of the value of $I(\gamma, z_0)$, the Residue Theorem gives

$$\oint_\gamma f(z)\, dz = 0.$$

The Residue Theorem has Cauchy's Integral formula also as special case.

When $f : U \to X$ is holomorphic, and $z_0 \in U$, then the function $g(z) = f(z)/(z - z_0)$ is holomorphic on $U \setminus \{z_0\}$, so for any simple closed curve γ in U enclosing z_0 the Residue Theorem gives

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\ dz = \frac{1}{2\pi i} \oint_\gamma g(z)\ dz = \mathrm{Res}(g, z_0) I(\gamma, z_0);$$

here $I(\gamma, z_0) = 1$ because γ is a simple closed curve enclosing z_0, and $\mathrm{Res}(g, z_0) = f(z_0)$ because using the power series for f about z_0 gives the Laurent series

$$g(z) = \frac{f(z)}{z - z_0} = \frac{1}{z - z_0} \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^{k-1}$$

in which the coefficient of $(z - z_0)^{-1}$ is $f(z_0)$.

What questions do you have?

To use the Residue Theorem requires that we compute the required residues.

We have seen two ways to compute the residue of f at a point z_0: by computing the Laurent series of f on $B(z_0, \epsilon) \setminus \{z_0\}$, or by Proposition 11.7.8 part (iii).

Of the many other means of computing $\mathrm{Res}(f, z_0)$ we mention another one.

Proposition 11.7.15. Suppose $g : B(z_0, \epsilon) \to X$ and $h : B(z_0, \epsilon) \to \mathbb{C}$ are holomorphic. If $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$, then the function $g(z)/h(z) : B(z_0, \epsilon) \setminus \{z_0\} \to X$ is meromorphic with a simple pole at z_0 and

$$\mathrm{Res}\left(\frac{g(z)}{h(z)}, z_0\right) = \frac{g(z_0)}{h'(z_0)}.$$

Note. While you are responsible for knowing and using Proposition 11.7.15, you are **NOT** responsible for the next proposition on computing the residue for a pole of order 2.

It is given to show you how complicated residue calculations can become for nonsimple poles.

Proposition. Suppose $g : B(z_0, \epsilon) \to X$ and $h : B(z_0, \epsilon) \to \mathbb{C}$ are holomorphic. If $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) = 0$, and $h''(z_0) \neq 0$, then $g(z)/h(z) : B(z_0, \epsilon) \setminus \{z_0\} \to X$ is meromorphic with a pole of order 2 at z_0, and

$$\operatorname{Res}\left(\frac{g(z)}{h(z)}, z_0\right) = \frac{2g'(z_0)}{h''(z_0)} - \frac{2g(z_0)h^{(3)}(z_0)}{3[h''(z_0)]^2}.$$

REMEMBER YOU ARE NOT RESPONSIBLE FOR THIS FORMULA.

Remark. A truly hideous formula for the residue of a pole of order N is given in my lecture notes.

It involves symbolic cofactor expansion of an $N \times N$ matrix.

Need I say more????

Remark. Unfortunately for an essential singularity of f at z_0 there are no "simple" formulas for computing the residue of f at z_0.

We typically rely on computing, somehow, the Laurent series for f at z_0 to find its residue at z_0.

Example (in lieu of 11.7.16). For the holomorphic function

$$f(z) = 1/(z^2 + 1)$$

the numerator is $g(z) = 1$ and the denominator is $h(z) = z^2 + 1$.

The roots of $h(z) = (z - i)(z + i)$ are $z_1 = i$ and $z_2 = -i$, i.e., $h(z_1) = 0$ and $h(z_2) = 0$.

Since $h'(z) = 2z$ we have

$$h'(z_1) = 2i \neq 0 \text{ and } h'(z_2) = -2i \neq 0.$$

By Proposition 11.7.15, the function f has a simple pole at each of z_1 and z_2 where

$$\mathrm{Res}(f, z_1) = \frac{g(z_1)}{h'(z_1)} = \frac{1}{2i} \text{ and } \mathrm{Res}(f, z_2) = \frac{g(z_2)}{h'(z_2)} = -\frac{1}{2i}.$$

The simple closed contour $\gamma = \{z \in \mathbb{C} : |z| = 2\}$, i.e., the circle centered at 0 with radius 2, encloses both simple poles of f. [Draw the picture]

For the winding numbers we have $I(\gamma, z_1) = 1$ and $I(\gamma, z_2) = 1$.

By the Residue Theorem we compute

$$\frac{1}{2\pi i} \oint_\gamma f(z)\ dz = \sum_{j=1}^{2} \mathrm{Res}(f, z_j) I(\gamma, z_j) = \frac{1}{2i} - \frac{1}{2i} = 0.$$

What questions do you have?

Example 11.7.17. Compute

$$\int_{-\infty}^{\infty} f(x)\ dx \text{ for } f(x) = \frac{1}{1+x^4}.$$

The improper integral of f over $\mathbb{R}$ converges by a comparison test with $1/(1+x^2)$, i.e., since $1+x^4 \geq 1+x^2$, then

$$0 \leq \frac{1}{1+x^4} \leq \frac{1}{1+x^2}$$

and the improper integral of $1/(1+x^2)$ converges because

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\ dx = \lim_{R\to\infty} \arctan(x)\Big|_{-R}^{R} = \pi < \infty.$$

Convergence of the improper integral of $1/(1+x^4)$ over $\mathbb{R}$ justifies writing

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\ dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\ dx.$$

We recognize that the integrand is equal to the complex-valued function

$$f(z) = \frac{1}{1+z^4} \text{ when } z \in \mathbb{R}.$$

The function $f(z)$ is complex differentiable except at the four roots of the denominator $h(z) = 1 + z^4$.

We can find these four roots using Euler's Formula as follows.

By writing

$$-1 = e^{i\pi + 2in\pi}$$

for an arbitrary integer n, the equation

$$1 + z^4 = 0 \text{ becomes } e^{i\pi+2in\pi} = z^4.$$

Taking fourth roots of both sides of this equation gives

$$e^{i\pi/4+ni\pi/2} = z.$$

The root complex roots of $h(z) = z^4 + 1$ are correspond to the four distinct angles $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$ in $[0, 2\pi)$; the four roots are

$$z_1 = e^{i\pi/4}, z_2 = e^{3i\pi/4}, z_3 = e^{5i\pi/4}, z_4 = e^{7i\pi/4}.$$

There is one root in each quadrant of the complex plane.

The function f is meromorphic on $\mathbb{C} \setminus \{z_1, z_2, z_3, z_4\}$.

Since $h'(z) = 4z^3$ and $h'(z_j) \neq 0$ for all $j = 1, 2, 3, 4$, each point z_j is a simple pole for $f(z) = 1/h(z)$ with residue

$$\mathrm{Res}(f, z_j) = \frac{1}{h'(z_j)} = \frac{1}{4z_j^3}.$$

Now for the "magic" of the Residue Theorem.

For $R \geq 2$, form the closed simple contour D that is the sum of the line γ from $-R$ to R and the top half C of the circle with center 0 and radius R traversed counterclockwise.
[Draw the picture]

This gives

$$\oint_D f(z)dz = \int_\gamma f(z)\, dz + \int_C f(z)\, dz = \int_{-R}^{R} \frac{1}{1+x^4}\, dx + \int_C f(z)\, dz.$$

The contour D encloses two simple poles of $f(z)$, the two in the first and second quadrant.

The residues of f at these poles are

$$\mathrm{Res}(f, z_1) = \frac{1}{4(e^{i\pi/4})^3} = \frac{1}{4e^{3i\pi/4}}$$
$$\mathrm{Res}(f, z_2) = \frac{1}{4(e^{3i\pi/4})^3} = \frac{1}{4e^{9i\pi/4}} = \frac{1}{4e^{i\pi/4}}.$$

The winding numbers of D at the poles are

$$I(D, z_j) = 1 \text{ for } j = 1, 2.$$

By the Residue Theorem we have

$$
\begin{aligned}
\oint_D \frac{1}{1+z^4}\, dz &= 2\pi i \left[\mathrm{Res}\left(\frac{1}{1+z^4}, z_1\right) + \mathrm{Res}\left(\frac{1}{1+z^4}, z_2\right)\right] \\
&= 2\pi i \left[\frac{1}{4e^{3i\pi/4}} + \frac{1}{4e^{i\pi/4}}\right] \\
&= \frac{\pi i}{2}\left[e^{-3i\pi/4} + e^{-i\pi/4}\right] \\
&= \frac{\pi i}{2}\left[\cos(3\pi/4) - i\sin(3\pi/4) + \cos(\pi/4) - i\sin(\pi/4)\right] \\
&= \frac{\pi i}{2}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] \\
&= \frac{\pi i}{2}\left(-\frac{2i}{\sqrt{2}}\right) \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}
$$

By the parameterization $\xi(\theta) = Re^{i\theta}$, $\theta \in [0, \pi]$, of C we obtain

$$\begin{aligned}
\left| \int_C \frac{1}{1+z^4} \, dz \right| &= \left| \int_0^\pi \frac{iRe^{i\theta}}{1+R^4e^{4i\theta}} \, d\theta \right| \\
&\leq \int_0^\pi \left| \frac{iRe^{i\theta}}{1+R^4e^{4i\theta}} \right| \, d\theta \\
&= \int_0^\pi \frac{R}{|1+R^4e^{4i\theta}|} \, d\theta \\
&\leq \int_0^\pi \frac{R}{|R^4e^{4i\theta}|-1} \, d\theta \\
&= \frac{R\pi}{R^4-1},
\end{aligned}$$

where for the last inequality we have used the "reverse" triangle inequality

$$|R^4e^{4i\theta}| - |-1| \leq |R^4e^{4i\theta} - (-1)|.$$

Letting $R \to \infty$ we obtain from

$$\oint_D f(z)dz = \int_{-R}^{R} \frac{1}{1+x^4}\,dx + \int_C f(z)\,dz$$

that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \oint_D f(z)dz = \frac{\pi}{\sqrt{2}}$$

since

$$\lim_{R\to\infty} \int_C f(z)\,dz = 0.$$

What questions do you have?