12.4 Spectral Resolution

April 3, 2020

Recall for a linear operator $A \in M_n(\mathbb{C})$ its resolvent $R_A : \rho(A) \to M_n(A)$ is defined by

$$R_A(z) = (zI - A)^{-1}, \; z \in \rho(A) = \mathbb{C} \setminus \sigma(A).$$

Definition 12.4.1. For $A \in M_n(\mathbb{C})$, let $\lambda \in \sigma(A)$ and Γ be a positively oriented simple closed curve enclosing λ but no other elements of $\sigma(A)$. The spectral projection or eigenprojection of A associated with λ is defined to be

$$P_\lambda = \mathrm{Res}(R_A(z), \lambda) = \frac{1}{2\pi i} \oint_\Gamma R_A(z) \, dz.$$

[Draw the picture.]

First Reading Quiz Question: What properties does the spectral projection have?

Theorem 12.4.2. For $A \in M_n(\mathbb{C})$, the spectral projections P_λ, $\lambda \in \sigma(A)$, have the following properties.

(i) Idempotency: $P_\lambda^2 = P_\lambda$ for all $\lambda \in \sigma(A)$.

(ii) Independence: $P_\lambda P_{\lambda'} = 0 = P_{\lambda'} P_\lambda$ for all $\lambda, \lambda' \in \sigma(A)$ with $\lambda \neq \lambda'$.

(iii) A-invariance: $AP_\lambda = P_\lambda A$ for all $\lambda \in \sigma(A)$ ("commutes"),

(iv) Completeness: $\sum_{\lambda \in \sigma(A)} P_\lambda = I$.

Note. Property (iii) $AP_\lambda = P_\lambda A$ implies the A-invariance of $\mathscr{R}(P_\lambda)$.

Proof. For $\mathrm{v} \in \mathscr{R}(P_\lambda)$ there exists $\mathrm{x} \in \mathbb{C}^n$ such that $\mathrm{v} = P_\lambda \mathrm{x}$.

Then $P_\lambda(A\mathrm{v}) = AP_\lambda \mathrm{v} = AP_\lambda P_\lambda \mathrm{x} = AP_\lambda \mathrm{x} = A\mathrm{v}$.

By Lemma 12.1.3, $A\mathrm{v}$ belongs to $\mathscr{R}(P_\lambda)$, i.e., $\mathscr{R}(P_\lambda)$ is A-invariant.

What questions do you have?

Example (in lieu of 12.4.4). Find the spectral projections for

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

Recall that the resolvent for this matrix is

$$R_A(z) = \frac{1}{(z-3)(z+1)} \begin{bmatrix} z-1 & 1 \\ 4 & z-1 \end{bmatrix}.$$

The partial fraction decompositions for the entries of $R_A(z)$ are

$$(1,1)\ (2,2) \quad \frac{z-1}{(z-3)(z+1)} = \frac{1/2}{z-3} + \frac{1/2}{z+1},$$

$$(1,2) \quad \frac{1}{(z-3)(z+1)} = \frac{1/4}{z-3} + \frac{-1/4}{z+1},$$

$$(2,1) \quad \frac{4}{(z-3)(z+1)} = \frac{1}{z-3} + \frac{-1}{z+1}.$$

We obtain the partial fraction decomposition

$$R_A(z) = \frac{1}{z-3}\begin{bmatrix} 1/2 & 1/4 \\ 1 & 1/2 \end{bmatrix} + \frac{1}{z+1}\begin{bmatrix} 1/2 & -1/4 \\ -1 & 1/2 \end{bmatrix}.$$

Recall that we can express $1/(z-3)$ as a geometric series in $(z+1)$, and we can express $1/(z+1)$ as a geometric series in $(z-3)$, giving the Laurent series of $R_A(z)$ about $z=3$ and about about $z=-1$.

From the partial fraction decomposition of $R_A(z)$ we have the spectral projections,

$$P_3 = \mathrm{Res}(R_A(z), 3) = \begin{bmatrix} 1/2 & 1/4 \\ 1 & 1/2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix},$$

and

$$P_{-1} = \mathrm{Res}(R_A(z), -1) = \begin{bmatrix} 1/2 & -1/4 \\ -1 & 1/2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}.$$

These spectral projections P_3 and P_{-1} are precisely the projections

$$P_1 = \mathrm{r}_1\ell_1^{\mathrm{T}} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix},\ P_2 = \mathrm{r}_2\ell_2^{\mathrm{T}} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$$

(slight abuse of subscript notation for projections) we computed in Lecture #31 by way of right eigenvectors

$$\mathrm{r}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \mathrm{r}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

of A, and the left eigenvectors

$$\ell_1^{\mathrm{T}} = \frac{1}{4}\begin{bmatrix} 2 & 1 \end{bmatrix},\ \ell_2^{\mathrm{T}} = \frac{1}{4} = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

of A that satisfy $\ell_i^{\mathrm{T}}\mathrm{r}_j = \delta_{ij}$.

We already have seen that these projections are idempotent, independent, commute with A, and sum to I.

What questions do you have?

Example. Find the spectral projections for

$$A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix}.$$

Recall from Lecture #33 that the resolvent for this matrix is

$$R_A(z) = \begin{bmatrix} (z-6)^{-1} & (z-6)^{-2} & 7(z-6)^{-2}(z-4)^{-1} \\ 0 & (z-6)^{-1} & 7(z-6)^{-1}(z-4)^{-1} \\ 0 & 0 & (z-4)^{-1} \end{bmatrix}.$$

The needed partial fraction decompositions are

$$\frac{7}{(z-6)^2(z-4)} = \frac{-7/4}{z-6} + \frac{7/2}{(z-6)^2} + \frac{7/4}{(z-4)},$$

$$\frac{7}{(z-6)(z-4)} = \frac{7/2}{z-6} + \frac{-7/2}{z-4}.$$

The partial fraction decomposition for $R_A(z)$ is

$$\frac{1}{z-6}\begin{bmatrix}1 & 0 & -7/4\\ 0 & 1 & 7/2\\ 0 & 0 & 0\end{bmatrix}+\frac{1}{(z-6)^2}\begin{bmatrix}0 & 1 & 7/2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\frac{1}{z-4}\begin{bmatrix}0 & 0 & 7/4\\ 0 & 0 & -7/2\\ 0 & 0 & 1\end{bmatrix}.$$

From this we obtain

$$P_6 = \mathrm{Res}(R_A(z),6) = \begin{bmatrix}1 & 0 & -7/4\\ 0 & 1 & 7/2\\ 0 & 0 & 0\end{bmatrix},$$

and

$$P_4 = \mathrm{Res}(R_A(z),4) = \begin{bmatrix}0 & 0 & 7/4\\ 0 & 0 & -7/2\\ 0 & 0 & 1\end{bmatrix}.$$

We check the four properties of Theorem 12.4.2 for P_6 and P_4.

The matrices P_6 and P_4 are idempotent:

$$P_6^2 = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = P_6,$$

and

$$P_4^2 = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} = P_4.$$

The matrices P_6 and P_4 are independent:

$$P_6 P_4 = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} = 0,$$

and

$$P_4 P_6 = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

The matrices P_6 and P_4 commute with A:

$$P_6 A = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -7 \\ 0 & 6 & 21 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A P_6 = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -7 \\ 0 & 6 & 21 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$P_4 A = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 7 \\ 0 & 0 & -14 \\ 0 & 0 & 4 \end{bmatrix},$$

$$A P_4 = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 7 \\ 0 & 0 & -14 \\ 0 & 0 & 4 \end{bmatrix}.$$

The matrices P_6 and P_4 are complete:

$$\underset{P_6}{\begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix}} + \underset{P_4}{\begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix}} = I.$$

Does $6P_6 + 4P_4 = A$ equal? No because

$$6P_6 + 4P_4 = \begin{bmatrix} 6 & 0 & -7/2 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix} \neq A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix}.$$

This should not be surprising since A is not diagonalizable.

But notice that

$$A - 6P_6 - 4P_4 = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Have we seen this matrix before?

Yes, we saw this matrix,

$$A - 6P_6 - 4P_4 = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in the resolvent $R_A(z)$,

$$\frac{1}{z-6}\begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{(z-6)^2}\begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{z-4}\begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix},$$

as the coefficient matrix of the $(z-6)^{-2}$ term.

Is this just a coincidence? We shall see.

What questions do you have?

Example (in lieu of 12.4.5). Find the spectral projections for

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Recall that we computed the resolvent of this matrix in Lecture #33, where we obtained

$$\det(zI - A) = (z-2)^3(z-5).$$

and $\mathrm{adj}(zI - A)$ is

$$\begin{bmatrix} (z-2)^2(z-5) & (z-2)(z-5) & z-5 & 3 \\ 0 & (z-2)^2(z-5) & (z-2)(z-5) & 3(z-2) \\ 0 & 0 & (z-2)^2(z-5) & 3(z-2)^2 \\ 0 & 0 & 0 & (z-2)^3 \end{bmatrix}.$$

The resolvent $R_A(z)$ is

$$\begin{bmatrix} (z-2)^{-1} & (z-2)^{-2} & (z-2)^{-3} & 3(z-2)^{-3}(z-5)^{-1} \\ 0 & (z-2)^{-1} & (z-2)^{-2} & 3(z-2)^{-2}(z-5)^{-1} \\ 0 & 0 & (z-2)^{-1} & 3(z-2)^{-1}(z-5)^{-1} \\ 0 & 0 & 0 & (z-5)^{-1} \end{bmatrix}.$$

The required partial fraction decompositions are

$$\frac{3}{(z-2)^3(z-5)} = \frac{-1/9}{z-2} + \frac{-1/3}{(z-2)^2} + \frac{-1}{(z-2)^3} + \frac{1/9}{z-5},$$

$$\frac{3}{(z-2)^2(z-5)} = \frac{-1/3}{z-2} + \frac{-1}{(z-2)^2} + \frac{1/3}{z-5},$$

$$\frac{3}{(z-2)(z-5)} = \frac{-1}{z-2} + \frac{1}{z-5}.$$

The partial fraction decomposition for the resolvent $R_A(z)$ is

$$\frac{1}{z-2}\underbrace{\begin{bmatrix}1 & 0 & 0 & -1/9\\ 0 & 1 & 0 & -1/3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}} + \frac{1}{(z-2)^2}\underbrace{\begin{bmatrix}0 & 1 & 0 & -1/3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}}$$

$$+\frac{1}{(z-2)^3}\underbrace{\begin{bmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}}$$

$$+\frac{1}{z-5}\underbrace{\begin{bmatrix}0 & 0 & 0 & 1/9\\ 0 & 0 & 0 & 1/3\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1\end{bmatrix}}.$$

From the partial fraction decomposition of the resolvent we obtain the spectral decompositions

$$P_2 = \mathrm{Res}(R_A(z), 2) = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$P_5 = \mathrm{Res}(R_A(z), 5) = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We verify the four properties of Theorem 12.4.2 for the matrices P_2 and P_5.

The matrices P_2 and P_5 are idempotent:

$$P_2^2 = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = P_2,$$

and

$$P_5^2 = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = P_5.$$

The matrices P_2 and P_5 are independent:

$$P_2 P_5 = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 0$$

and

$$P_5 P_2 = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0.$$

The matrices P_2 and P_5 commute with A:

$$AP_2 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & -5/9 \\ 0 & 2 & 1 & -5/3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_2A = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & -5/9 \\ 0 & 2 & 1 & -5/3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$AP_5 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 5/9 \\ 0 & 0 & 0 & 5/3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

$$P_5A = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 5/9 \\ 0 & 0 & 0 & 5/3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

The matrices P_2 and P_5 are complete:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{P_2} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{P_5} = I.$$

Since A is not diagonalizable,

$$2P_2 + 5P_5 \neq A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Something else is needed that comes from the resolvent $R_A(z)$, something that will be fully explored in the next two sections.

For now we learn a few other things in preparation.

What questions do you have?

Theorem 12.4.6 (Spectral Resolution Theorem). For $A \in M_n(\mathbb{C})$, if the power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

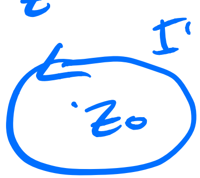
has a radius of convergence $b > r(A)$, then for any positively oriented simple closed contour Γ containing $\sigma(A)$ and contained within the disk $B(0, b_0)$ for some $b_0 \in (r(A), b)$, there holds

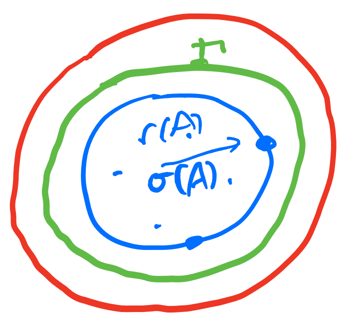
$$\sum_{k=0}^{\infty} a_k A^k = f(A) = \frac{1}{2\pi i} \oint_\Gamma f(z) R_A(z)\, dz.$$

$(zI - A)^{-1}$

[Draw the picture]

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz$$





Proof. WLOG we can assume that Γ is the circle centered at 0 with radius b_0. Using the Laurent series for $R_A(z)$ on $|z| > r(A)$, we have

$$\frac{1}{2\pi i} \oint_\Gamma f(z) R_A(z)\, dz = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z} \sum_{k=0}^{\infty} \frac{A^k}{z^k}\, dz.$$

Since $f(z)/z$ is bounded on Γ and the summation converges uniformly on compact sets, the sum and integral can be interchanged to give

$$\frac{1}{2\pi i} \oint_\Gamma f(z) R_A(z)\, dz = \sum_{k=0}^{\infty} A^k \left[\frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z^{k+1}}\, dz \right].$$

By Cauchy's Differentiation formula we have

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z^{k+1}}\, dz.$$

Thus we obtain

$$\frac{1}{2\pi i} \oint_\Gamma f(z) R_A(z)\, dz = \sum_{k=0}^{\infty} a_k A^k = f(A).$$

Second Reading Quiz Question: The spectral radius $r(A)$ is the supremum of the absolute values of the eigenvalues of A.

True

Corollary 12.4.7. The spectral radius of $A \in M_n(\mathbb{C})$ satisfies

$$\boxed{r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}.}$$

Note. For a polynomial $q(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \in \mathbb{C}[z]$ and $A \in M_n(\mathbb{C})$, recall that

$$\boxed{q(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n.}$$

Corollary 12.4.8 (Cayley-Hamilton Theorem). For $A \in M_n(\mathbb{C})$, let $p(z) = \det(zI - A)$, the characteristic polynomial of A. Then $p(A) = 0$.

Nota Bene 12.4.9. What this poorly worded remark wanted to say is that no polynomial in A is formed if we replace z by A on the right side of $p(z) = \det(zI - A)$ and then compute the determinant.

A

Example (in lieu of 12.4.10). We verify the Cayley-Hamilton Theorem for

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

The characteristic polynomial of A is

$$p(z) = (z-3)(z+1) = z^2 - 2z - 3. \qquad 3I$$

Replacing z by A in the characteristic polynomial we have

$$\begin{aligned} A^2 - 2A - 3I &= \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 2 \\ 8 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 8 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

What questions do you have?

$$r(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$$

independent of $\|\cdot\|$

by Carollary 12.4.7.