12.6 Spectral Decomposition II

April 8, 2020

We are now in the position of proving, for every linear operator $A \in M_n(\mathbb{C})$, the existence of the spectral decomposition

$$A = \sum_{\lambda \in \sigma(A)} \left(\lambda P_\lambda + D_\lambda \right)$$

where

- P_λ is the matrix coefficient A_{-1} in the Laurent series of $R_A(z)$ about λ,
- $\mathscr{R}(P_\lambda) = \mathscr{E}_\lambda$, the generalized eigenspace of A for λ, and
- D_λ is the matrix coefficient A_{-2} in the Laurent series of $R_A(z)$ about λ.

We have seen this spectral decomposition in two examples already.

The computation tool for finding the spectral decomposition of A is the method of partial fractions applied to the nonzero entries of the resolvent

$$R_A(z) = (zI - A)^{-1}.$$

Yes, it is that straightforward.

Lemma 12.6.1. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, the linear operator $D_\lambda \in M_n(\mathbb{C})$ satisfies

$$D_\lambda = (A - \lambda I)P_\lambda.$$

Moreover, the spectral radius of D_λ is zero, i.e., $r(D_\lambda) = 0$.

Note. The equation

$$D_\lambda = (A - \lambda I)P_\lambda$$

holds if and only if

$$\boxed{AP_\lambda = \lambda P_\lambda + D_\lambda}$$

holds.

Note. Since

$$r(D_\lambda) = \sup\{|\mu| : \mu \in \sigma(D_\lambda)\} = 0$$

the only eigenvalue of D_λ is 0 and it has an algebraic multiplicity of n.

Example (in lieu of 12.6.2). We verify

$$D_\lambda = (A - \lambda I)P_\lambda$$

for

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

and $\lambda = 2$.

Recall that

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the verification of

$$D_\lambda = (A - \lambda I)P_\lambda$$

for $\lambda = 2$ we have

$$\begin{aligned}(A-2I)P_2 &= \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & -1/9\\ 0 & 1 & 0 & -1/3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}0 & 1 & 0 & -1/3\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\\ &= D_2.\end{aligned}$$

This implies that

$$AP_2 = 2P_2 + D_2.$$

What questions do you have?

Remark. Recall that a matrix $B \in M_n(\mathbb{C})$ is called nilpotent if there is $l \in \mathbb{N}$ such that $B^l = 0$.

Lemma 12.6.3. A matrix $B \in M_n(\mathbb{C})$ satisfies $r(B) = 0$ if and only if B is nilpotent.

Proof. Suppose that $r(B) = 0$.

Then $\sigma(B) = \{0\}$, i.e., every eigenvalue of B is zero.

Hence the characteristic polynomial of B is $p(z) = z^n$.

By the Cayley-Hamilton Theorem, we have have $B^n = p(B) = 0$, which says that B is nilpotent.

Now suppose that B is nilpotent.

Then there exists $l \in \mathbb{N}$ such that $B^l = 0$.

This implies that $B^k = 0$ for all $k \geq l$.

Hence for any matrix norm $\|\cdot\|$ we have $\|B^k\| = 0$ for all $k \geq l$.

This implies that $r(B) = \lim_{k\to\infty} \|B^k\|^{1/k} = 0$.

Remark. Lemma 12.6.1,

$$r(D_\lambda) = 0,$$

and Lemma 12.6.3,

$$r(B) = 0 \ \Leftrightarrow \ B \text{ is nilpotent},$$

show that the linear operator

$$D_\lambda \text{ is nilpotent.}$$

Definition. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, the nilpotent linear operator D_λ is called the eigennilpotent of A associated with the eigenvalue λ.

Note. The presence of a nonzero D_λ in the spectral decomposition indicates the deviation of the linear operator from being semisimple.

Remark 12.6.4. Recall that the order of a nilpotent $B \in M_n(\mathbb{C})$ is the smallest $l \in \mathbb{N}$ such that $B^l = 0$.

For a nilpotent B of order l, each B^k with $0 \leq k < l$, we have $B^k \neq 0$ so that $\mathscr{N}(B^k)$ is a proper subspace of $\mathbb{C}^n$.

Since $B^l = B^{l+1} = 0$, then

$$\mathscr{N}(B^l) = \mathscr{N}(B^{l+1}) = \mathbb{C}^n,$$

and so $\mathrm{ind}(B) = l$, i.e., the order of B *is the same as the index of* B.

Second Reading Quiz Question: The index of a nilpotent matrix is the same as its order.

True

From Exercise 12.6, the index of a nilpotent B is no bigger than n, meaning that $\mathrm{ind}(B) \leq n$.

What questions do you have?

Proposition 12.6.5. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, the order m_λ of the eigennilpotent D_λ of A satisfies

$$m_\lambda \leq \dim(\mathscr{R}(P_\lambda)).$$

Proof. By Lemma 12.5.5 part (iii) we have $D_\lambda = P_\lambda D_\lambda = D_\lambda P_\lambda$.

We show (a) $\mathscr{R}(P_\lambda)$ is D_λ-invariant and (b) $\mathscr{R}(D_\lambda) \subset \mathscr{R}(P_\lambda)$.

To show (a) $\mathscr{R}(P_\lambda)$ is D_λ-invariant, let $\mathrm{y} \in \mathscr{R}(P_\lambda)$.

Then there exists $\mathrm{x} \in \mathbb{C}^n$ such that $\mathrm{y} = P_\lambda \mathrm{x}$.

Hence $D_\lambda \mathrm{y} = D_\lambda P_\lambda \mathrm{x} = P_\lambda (D_\lambda \mathrm{x}) \in \mathscr{R}(P_\lambda) \Rightarrow \mathscr{R}(P_\lambda)$ is D_λ-invariant.

To show (b) $\mathscr{R}(D_\lambda) \subset \mathscr{R}(P_\lambda)$, let $\mathrm{y} \in \mathscr{R}(D_\lambda)$.

Then there is $\mathrm{x} \in \mathbb{C}^n$ such that $\mathrm{y} = D_\lambda \mathrm{x}$.

Hence $\mathrm{y} = D_\lambda \mathrm{x} = P_\lambda (D_\lambda \mathrm{x}) \in \mathscr{R}(P_\lambda) \Rightarrow \boxed{\mathscr{R}(D_\lambda) \subset \mathscr{R}(P_\lambda)}$.

We can thus consider $D_\lambda = D_\lambda P_\lambda$ as a linear operator on $\mathscr{R}(P_\lambda)$.

That $m_\lambda \le \dim(\mathscr{R}(P_\lambda))$ now follows from Exercise 12.6, i.e., $\mathrm{ind}(D_\lambda) \le \dim(\mathscr{R}(P_\lambda))$. $\square$

Example. We illustrate $\mathscr{R}(D_\lambda) \subset \mathscr{R}(P_\lambda)$ for

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

for $\lambda = 2$ where

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here

$$\mathscr{R}(P_2) = \mathrm{span}(\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3) \supset \mathscr{R}(D_2) = \mathrm{span}(\mathrm{e}_1, \mathrm{e}_2).$$

Remark 12.6.6. Proposition 12.6.5 implies that the resolvent $R_A(z)$ has no essential singularities.

So $R_A(z)$ is meromorphic on $\rho(A)$.

More precisely, part (iv) of Lemma 12.5.5 simplifies to

$$R_A(z) = \frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}} + \sum_{k=0}^{\infty} (-1)^k (z-\lambda)^k S_\lambda^{k+1}$$

and part (v) of Lemma 12.5.5 simplifies to

$$\boxed{R_A(z)P_\lambda = \frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}}}.$$

Note. The isolated singularity of $R_A(z)$ at $z = \lambda$ is a simple pole if $D_\lambda = 0$ or is a pole of order $m_\lambda \geq 2$ if $D_\lambda \neq 0$.

What questions do you have?

Remark. We now turn attention to showing that

$$\mathscr{R}(P_\lambda) = \mathscr{E}_\lambda,$$

and developing some results to be used in the next section to establish uniqueness of the spectral decomposition.

We notice that if $\mathrm{y} \in \mathscr{R}(P_\lambda)$, then $(\lambda I - A)\mathrm{y} \in \mathscr{R}(P_\lambda)$ because for $\mathrm{y} = P_\lambda \mathrm{x}$ we have

$$(\lambda I - A)\mathrm{y} = (\lambda I - A)P_\lambda \mathrm{x} = D_\lambda(-\mathrm{x}) \in \mathscr{R}(D_\lambda),$$

where we are using

$$D_\lambda = (A - \lambda I)P_\lambda = -(\lambda I - A)P_\lambda,$$

and, as shown in the proof of Proposition 12.6.5, that $\mathscr{R}(D_\lambda) \subset \mathscr{R}(P_\lambda)$, whence that

$$(\lambda I - A)\mathrm{y} \in \mathscr{R}(P_\lambda).$$

The converse is also true as the next result asserts.

Lemma 12.6.7. For $A \in M_n(\mathbb{C})$, let $\lambda \in \sigma(A)$ and $\mathrm{y} \in \mathbb{C}^n$. If $(\lambda I - A)\mathrm{y} \in \mathscr{R}(P_\lambda)$, then $\mathrm{y} \in \mathscr{R}(P_\lambda)$.

Remark 12.6.8. The proof of Lemma 12.6.7 only depends on the following properties of the eignprojections P_μ and eigennilpotents D_μ:

(1) $\sum_{\mu \in \sigma(A)} P_\mu = I$,

(2) $P_\mu P_{\mu'} = 0$ for $\mu \neq \mu'$,

(3) $D_\mu P_\mu = D_\mu$, and

(4) $AP_\mu = \mu P_\mu + D_\mu$ for all $\mu \in \sigma(A)$.

This is important for the uniqueness of the spectral decomposition.

Theorem 12.6.9. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, there holds

$$\mathscr{R}(P_\lambda) = \mathscr{E}_\lambda.$$

Idea of Proof. By repeated use of Lemma 12.6.7,

$$(\lambda I - A)\mathrm{y} \in \mathscr{R}(P_\lambda) \Rightarrow \mathrm{y} \in \mathscr{R}(P_\lambda),$$

we obtain $\mathscr{E}_\lambda \subset \mathscr{R}(P_\lambda)$ as follows.

Recall that $\mathscr{E}_\lambda = \mathscr{N}((\lambda I - A)^{k_\lambda})$ where $k_\lambda = \mathrm{ind}(\lambda I - A)$.

For $\mathrm{y} \in \mathscr{N}((\lambda I - A)^{k_\lambda})$, we have

$$(\lambda I - A)((\lambda I - A)^{k_\lambda - 1}\mathrm{y}) = (\lambda I - A)^{k_\lambda}\mathrm{y} = 0 \in \mathscr{R}(P_\lambda),$$

so that

$$(\lambda I - A)^{k_\lambda - 1}\mathrm{y} \in \mathscr{R}(P_\lambda).$$

Continue until we reach

$$(\lambda I - A)\mathrm{y} \in \mathscr{R}(P_\lambda) \Rightarrow \mathrm{y} \in \mathscr{R}(P_\lambda).$$

This gives $\mathscr{E}_\lambda \subset P_\lambda$.

To get $\mathscr{E}_\lambda = \mathscr{R}(P_\lambda)$ for all $\lambda \in \sigma(A)$, the idea is to establish

$$\mathbb{C}^n = \bigoplus_{\lambda \in \sigma(A)} \mathscr{R}(P_\lambda)$$

and compare this with the already established

$$\mathbb{C}^n = \bigoplus_{\lambda \in \sigma(A)} \mathscr{E}_\lambda.$$

The inclusions $\mathscr{E}_\lambda \subset P_\lambda$ force the equalities $\mathscr{E}_\lambda = \mathscr{R}(P_\lambda)$ for all $\lambda \in \sigma(A)$.

The direct sum decomposition with $\mathscr{R}(P_\lambda)$ is established using their completeness and independence.

Remark 12.6.10. The proof of Theorem 12.6.9 only depends properties of the projections listed in Remark 12.6.8. This is important in the next section when we prove the uniqueness of the spectral decomposition.

What questions do you have?

Theorem 12.6.12 (Spectral Decomposition Theorem). For $A \in M_n(\mathbb{C})$, and $\lambda \in \sigma(A)$, let P_λ be the spectral projection of A associated to λ, and let D_λ be the eigennilpotent of A associated to λ with its order m_λ. The resolvent of A takes the form

$$R_A(z) = \sum_{\lambda \in \sigma(A)} \left[\frac{P_\lambda}{z - \lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}} \right],$$

and there holds the spectral decomposition

$$\boxed{A = \sum_{\lambda \in \sigma(A)} \left(\lambda P_\lambda + D_\lambda \right)}.$$

Proof. From Lemma 12.5.5 part (v) and the nilpotency of D_λ we have

$$R_A(z) P_\lambda = \frac{P_\lambda}{z - \lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}}.$$

Combining this with the completeness of the spectral projections gives

$$\begin{aligned} R_A(z) &= R_A(z)I \\ &= R_A(z) \sum_{\lambda \in \sigma(A)} P_\lambda \\ &= \sum_{\lambda \in \sigma(A)} R_A(z) P_\lambda \\ &= \sum_{\lambda \in \sigma(A)} \left[\frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}} \right]. \end{aligned}$$

Recall that $AP_\lambda = \lambda P_\lambda + D_\lambda$.

Combining this with the completeness of the spectral projections gives

$$A = AI = A \sum_{\lambda \in \sigma(A)} P_\lambda = \sum_{\lambda \in \sigma(A)} AP_\lambda = \sum_{\lambda \in \sigma(A)} \big(\lambda P_\lambda + D_\lambda\big).$$

First Reading Quiz Question: Comment on the relationship between the form of the resolvent presented in the Spectral Decomposition Theorem 12.6.12,

$$\boxed{R_A(z) = \sum_{\lambda \in \sigma(A)} \left[\frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}} \right]},$$

and the partial fraction decomposition of the resolvent.

Remark. The form of the resolvent stated in the Spectral Decomposition Theorem is the **PRECISELY** form we have already been getting by using the partial fraction decompositions for the rational function entries of the resolvent.

What questions do you have?

Example (in lieu of 12.6.13). Find the spectral decomposition for the linear operator

$$A = \begin{bmatrix} -1 & 11 & -3 \\ -2 & 8 & -1 \\ -1 & 5 & 0 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\det(zI - A) = z^3 - 7z^2 + 16z - 12 = (z-2)^2(z-3).$$

The adjugate of $zI - A$ is

$$\operatorname{adj}(zI - A) = \begin{bmatrix} z^2 - 8z + 5 & 11z - 15 & -3z + 13 \\ -2z + 1 & z^2 + z - 3 & -z + 5 \\ -z - 2 & 5z - 6 & z^2 - 7z + 14 \end{bmatrix}.$$

Performing nine partial fraction decompositions (one for each entry) the resolvent is

$$\frac{1}{z-2}\begin{bmatrix} 11 & -18 & -4 \\ 5 & -8 & -2 \\ 5 & -9 & -1 \end{bmatrix} + \frac{1}{(z-2)^2}\begin{bmatrix} 7 & -7 & -7 \\ 3 & -3 & -3 \\ 4 & -4 & -4 \end{bmatrix} + \frac{1}{z-3}\begin{bmatrix} -10 & 18 & 4 \\ -5 & 9 & 2 \\ -5 & 9 & 2 \end{bmatrix}$$

From this we have

$$P_2 = \begin{bmatrix} 11 & -18 & -4 \\ 5 & -8 & -2 \\ 5 & -9 & -1 \end{bmatrix}, \; D_2 = \begin{bmatrix} 7 & -7 & -7 \\ 3 & -3 & -3 \\ 4 & -4 & -4 \end{bmatrix}, \; P_3 = \begin{bmatrix} -10 & 18 & 4 \\ -5 & 9 & 2 \\ -5 & 9 & 2 \end{bmatrix},$$

whence the spectral decomposition is

$$A = 2P_2 + D_2 + 3P_3.$$

From this spectral decomposition we can finding "quicker" means of computing powers of A, such as

$$\begin{aligned} A^2 &= (2P_2 + D_2 + 3P_3)(2P_2 + D_2 + 3P_3) \\ &= 4P_2^2 + 2P_2D_2 + 6P_2P_3 + 2D_2P_2 + D_2^2 + 3D_2P_3 \\ &\quad + 6P_3P_2 + 3P_3D_2 + 9P_3^2 \\ &= 4P_2 + 4D_2 + 9P_3. \end{aligned}$$

Not only can we take powers of A, we can also take holomorphic images of A, and get expressions that look an awful lot like spectral decompositions!

Corollary 12.6.14. For $A \in M_n(\mathbb{C})$, let f be holomorphic complex valued function defined on a simply connected open set containing $\sigma(A)$. If for $\lambda \in \sigma(A)$, the complex constants $a_{n,\lambda}$ are the coefficients in the power series expansion of f about λ, i.e.,

$$f(z) = f(\lambda) + \sum_{n=1}^{\infty} a_{n,\lambda}(z-\lambda)^n,$$

then

$$f(A) = \sum_{\lambda \in \sigma(A)} \left[f(\lambda)P_\lambda + \sum_{k=1}^{m_\lambda - 1} a_{k,\lambda} D_\lambda^k \right].$$

In the case that A is semisimple the expression simplifies to

$$f(A) = \sum_{\lambda \in \sigma(A)} f(\lambda)P_\lambda.$$

Example (in lieu of 12.6.15). In the previous example, we used the spectral decomposition

$$A = 2P_2 + D_2 + 3P_3$$

to directly compute

$$A^2 = 4P_2 + 4D_2 + 9P_3.$$

We will use Corollary 12.6.14 to compute this by finding the coefficients of the power series expansion of the square function expanded about $\lambda = 2$:

$$f(z) = z^2 = (z-2+2)^2 = ((z-2)+2)^2 = 4+4(z-2)+(z-2)^2.$$

The Taylor series coefficients of $f(z)$ about $\lambda = 2$ are

$$a_{0,2} = 4,\ a_{1,2} = 4,\ a_{2,2} = 1,\ a_{k,2} = 0 \text{ for all } k \geq 3.$$

Since $D_3 = 0$ we do not need the Taylor coefficients of $f(z) = z^2$ expanded about $\lambda = 3$.

By Corollary 12.6.14, using $a_{1,2} = 4$, we have

$$\begin{aligned} A^2 &= f(A) \\ &= \sum_{\lambda \in \sigma(A)} \left[f(\lambda) P_\lambda + \sum_{k=1}^{m_\lambda - 1} a_{k,\lambda} D_\lambda^k \right] \\ &= 2^2 P_2 + 4 D_2 + 3^2 P_3 \\ &= 4 P_2 + 4 D_2 + 9 P_3. \end{aligned}$$

This agrees with what we computed earlier, but this is much faster.

What questions do you have?