Math 346 Lecture #7
7.1 Contraction Mapping Principle

Definition 7.1.1. For a nonempty set X and a function $f : X \to X$, a point $\bar{x} \in X$ is called a fixed point of f if $f(\bar{x}) = \bar{x}$.

Recall the One-Dimensional Brouwer Fixed Point Theorem: if $f : [0,1] \to [0,1]$ is continuous, then there exists $\bar{x} \in [0,1]$ such that $f(\bar{x}) = \bar{x}$, i.e., f has a fixed point, although it may have more than one fixed point.

There are other kinds of continuous functions that have fixed points.

Definition 7.1.3. Let $(X, \|\cdot\|)$ be a normed linear space, and D a nonempty subset of X. A function $f : D \to D$ is called a contraction mapping if there exists $0 \le k < 1$ such that for all $\mathrm{x}, \mathrm{y} \in D$ there holds

$$\|f(\mathrm{x}) - f(\mathrm{y})\| \le k\|\mathrm{x} - \mathrm{y}\|.$$

Remark 7.1.4. A contraction mapping $f : D \to D$ is Lipschitz continuous on D with constant k, and hence continuous on D.

Example 7.1.6. For $\mu \in [0,1]$ define $T : [0,1] \to [0,1]$ by

$$T(x) = 4\mu x(1-x).$$

That the codomain of T is correct for each $\mu \in [0,1]$ follows from the unique maximum of T occurring at $x = 1/2$ when $\mu > 0$, so that the maximum value of T is $4\mu(1/4) = \mu$.

The function T, known as the logistic map, is an iterative model in population dynamics, i.e., starting with a population percentage of x_0, the next population percentage is $x_1 = T(x_0)$, the next is $x_2 = T(x_1) = T^2(x_0)$, etc.

To show that T is contraction mapping for some μ, we start with inequality

$$|1 - x - y| \le 1$$

that holds for all $x, y \in [0,1]$. Then

$$\begin{aligned}
|T(x) - T(y)| &= |4\mu x(1-x) - 4\mu y(1-y)| \\
&= 4\mu|x - x^2 - y + y^2| \\
&= 4\mu|x - y - (x^2 - y^2)| \\
&= 4\mu|x - y - (x-y)(x+y)| \\
&= 4\mu|(x-y)(1-(x+y))| \\
&= 4\mu|(x-y)(1-x-y)| \\
&= 4\mu|x-y|\,|1-x-y| \\
&\le 4\mu|x-y|.
\end{aligned}$$

This implies that T is contraction mapping when $4\mu < 1$, i.e., when $\mu \in [0, 1/4)$.

Theorem 7.1.8 (Contraction Mapping Principle). For a nonempty closed subset D of a Banach space $(X, \|\cdot\|)$, if $f : D \to D$ is a contraction mapping, then there exists a unique fixed point $\bar{\mathrm{x}} \in D$ for f.

Proof. For a fixed $\mathrm{x}_0 \in D$ we form a sequence $(\mathrm{x}_n)_{n=0}^{\infty}$ iteratively by

$$\mathrm{x}_{n+1} = f(\mathrm{x}_n),\ n = 0, 1, 2, 3, \ldots.$$

We show that the sequence $(\mathrm{x}_n)_{n=0}^{\infty}$ is Cauchy in X.

Let $k \in [0, 1)$ be the constant for the contraction mapping $f : D \to D$, i.e.,

$$\|f(\mathrm{x}) - f(\mathrm{y})\| \leq k\|\mathrm{x} - \mathrm{y}\|$$

for all $\mathrm{x}, \mathrm{y} \in D$.

Then for $n \geq 2$ we have

$$\begin{aligned}
\|\mathrm{x}_n - \mathrm{x}_{n-1}\| &= \|f(\mathrm{x}_{n-1}) - f(\mathrm{x}_{n-2})\| \\
&\leq k\|\mathrm{x}_{n-1} - \mathrm{x}_{n-2}\| \\
&= k\|f(\mathrm{x}_{n-2}) - f(\mathrm{x}_{n-3})\| \\
&\leq k^2\|\mathrm{x}_{n-2} - \mathrm{x}_{n-3}\| \\
&\vdots \\
&= k^{n-2}\|\mathrm{x}_2 - \mathrm{x}_1\| \\
&= k^{n-2}\|f(\mathrm{x}_1) - f(\mathrm{x}_0)\| \\
&\leq k^{n-1}\|\mathrm{x}_1 - \mathrm{x}_0\|.
\end{aligned}$$

Hence for $n > m \geq 0$ we have

$$\begin{aligned}
\|\mathrm{x}_n - \mathrm{x}_m\| &= \|\mathrm{x}_n - \mathrm{x}_{n-1} + \mathrm{x}_{n-1} - \mathrm{x}_{n-2} + \cdots + \mathrm{x}_{m+2} - \mathrm{x}_{m+1} + \mathrm{x}_{m+1} - \mathrm{x}_m\| \\
&\leq \|\mathrm{x}_n - \mathrm{x}_{n-1}\| + \|\mathrm{x}_{n-1} - \mathrm{x}_{n-2}\| + \cdots + \|\mathrm{x}_{m+2} - \mathrm{x}_{m+1}\| + \|\mathrm{x}_{m+1} - \mathrm{x}_m\| \\
&\leq k^{n-1}\|\mathrm{x}_1 - \mathrm{x}_0\| + k^{n-2}\|\mathrm{x}_1 - \mathrm{x}_0\| + \cdots + k^{m+1}\|\mathrm{x}_1 - \mathrm{x}_0\| + k^m\|\mathrm{x}_1 - \mathrm{x}_0\| \\
&= k^m(1 + k + k^2 + \cdots + k^{n-m-1})\|\mathrm{x}_1 - \mathrm{x}_0\|.
\end{aligned}$$

We get an upper bound on $1+k+k^2+\cdots+k^{n-m-1}$ using the geometric series in $k \in [0, 1)$, i.e.,

$$\sum_{l=0}^{\infty} k^l = \frac{1}{1-k}$$

and that $k \geq 0$, so that

$$1 + k + k^2 + \cdots + k^{n-m-1} \leq \sum_{l=0}^{\infty} k^l = \frac{1}{1-k}.$$

Thus for $n > m \geq 0$ we have

$$\|\mathrm{x}_n - \mathrm{x}_m\| \leq \frac{k^m}{1-k}\|\mathrm{x}_1 - \mathrm{x}_0\|.$$

Since $k \in [0, 1)$, we have $k^m \to 0$ as $m \to \infty$.

To see that $(\mathrm{x}_n)_{n=0}^\infty$ is Cauchy in X, for $\epsilon > 0$ we choose $N \in \mathbb{N}$ so that

$$\frac{k^N}{1-k}\|\mathrm{x}_1 - \mathrm{x}_0\| < \epsilon,$$

for then

$$\|\mathrm{x}_n - \mathrm{x}_m\| \leq \frac{k^N}{1-k}\|\mathrm{x}_1 - \mathrm{x}_0\| < \epsilon.$$

The completeness of X implies that the Cauchy sequence $(\mathrm{x}_n)_{n=0}^\infty$ converges to an element $\bar{\mathrm{x}} \in X$.

Since D is closed and $(\mathrm{x}_n)_{n=0}^\infty \subset D$, the limit point $\bar{\mathrm{x}}$ belongs to D.

To see that $\bar{x}$ is a fixed point of f we have for $\epsilon > 0$ the existence of $N \in \mathbb{N}$ such that $\|\mathrm{x}_n - \bar{\mathrm{x}}\| < \epsilon/2$ for all $n \geq N$; hence for all $n \geq N$ we have

$$\begin{aligned}\|f(\bar{x}) - \bar{x}\| &= \|f(\bar{\mathrm{x}}) - f(\mathrm{x}_n) + f(\mathrm{x}_n) - \bar{\mathrm{x}}\| \\ &\leq \|f(\bar{\mathrm{x}}) - f(\mathrm{x}_n)\| + \|f(\mathrm{x}_n) - \bar{\mathrm{x}}\| \\ &\leq k\|\bar{\mathrm{x}} - \mathrm{x}_n\| + \|\mathrm{x}_{n+1} - \bar{\mathrm{x}}\| \\ &\leq \frac{k\epsilon}{2} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain $\|f(\bar{\mathrm{x}}) - \bar{\mathrm{x}}\| = 0$, i.e., $f(\bar{\mathrm{x}}) = \bar{\mathrm{x}}$.

To show the uniqueness of the fixed point $\bar{\mathrm{x}}$ suppose there was another fixed point $\bar{\mathrm{y}}$ of f in D.

Then

$$\|\bar{\mathrm{x}} - \bar{\mathrm{y}}\| = \|f(\bar{\mathrm{x}}) - f(\bar{\mathrm{y}})\| \leq k\|\bar{\mathrm{x}} - \bar{\mathrm{y}}\|.$$

This implies, since $0 \leq k < 1$ that $\|\bar{\mathrm{x}} - \bar{\mathrm{y}}\| = 0$, i.e., $\bar{\mathrm{x}} = \bar{\mathrm{y}}$. □

Remark 7.1.9. The contraction mapping principle also applies to contraction mappings $f : D \to D$ on a closed subset D of a complete metric space (X, d) by replacing every occurrence of $\|\mathrm{x} - \mathrm{y}\|$ with $d(\mathrm{x}, \mathrm{y})$, etc., in the definition of contraction mapping and the proof of the Contraction Mapping Principle.

Remark 7.1.10. The proof of the Contraction Mapping Principle uses a technique called the method of successive approximations for finding the fixed point.

No matter the initial guess x_0 for the fixed point, the sequence $(\mathrm{x}_n)_{n=1}^\infty$ determined iteratively by $\mathrm{x}_{n+1} = f(\mathrm{x}_n)$ always converges to the unique fixed point $\bar{\mathrm{x}}$ of f on D.

An bound on the error of the m^{th} approximation x_m for $\bar{\mathrm{x}}$ is obtained from the Cauchy estimate,

$$\|\bar{\mathrm{x}} - \mathrm{x}_m\| = \lim_{n\to\infty}\|\mathrm{x}_n - \mathrm{x}_m\| \leq \frac{k^m}{1-k}\|\mathrm{x}_1 - \mathrm{x}_0\|,$$

where we have pulled the limit outside the norm by its the continuity.

Vista 7.1.11. The contraction mapping principle is used to prove existence of solutions to many equations, such as initial value problems for systems of first-order differential equations (coming your way in Volume 4, if not sooner in disguise).

Example 7.1.12. An algorithm for finding the positive square roots of real numbers greater than one uses the method of successive approximations.

For $c^2 = b \geq 1$ the number $c \geq 1$ (the positive square root of c^2) is a fixed point of the function $f : (0, \infty) \to (0, \infty)$ defined by

$$f(x) = \frac{1}{2}\left(x + \frac{b}{x}\right)$$

because

$$f(c) = \frac{1}{2}\left(c + \frac{b}{c}\right) = \frac{1}{2}\left(c + \frac{c^2}{c}\right) = c.$$

The function f is not a contraction mapping on $(0, \infty)$: for $x = 2/n$ and $y = 1/n$ we have

$$f(x) - f(y) = \frac{1}{2}\left(\frac{2}{n} + \frac{bn}{2} - \frac{1}{n} - bn\right) = \frac{1}{2n} - \frac{bn}{4} = \frac{2}{4n} - \frac{bn^2}{4n} = \frac{2 - bn^2}{4n}$$

so that for all sufficiently large n we have

$$|f(2/n) - f(1/n)| = \frac{bn^2 - 2}{4n}$$

while

$$\left|\frac{2}{n} - \frac{1}{n}\right| = \frac{1}{n},$$

so that for all sufficiently large n we have

$$|f(2/n) - f(1/n)| = \frac{bn^2 - 4}{4n} > \frac{1}{n} = \left|\frac{2}{n} - \frac{1}{n}\right|.$$

But f is a contraction mapping on the closed interval $[\sqrt{b/2}, \infty)$.

To see this we start with

$$|f(x) - f(y)| = \frac{1}{2}\left|x - y + \frac{b}{x} - \frac{b}{y}\right| = \frac{1}{2}\left|x - y + \frac{b(y - x)}{xy}\right| = \frac{1}{2}\left|(x - y)\left(1 - \frac{b}{xy}\right)\right|.$$

When $x, y \geq \sqrt{b/2}$, then $xy \geq b/2$ and $-2 \leq -b/xy$, so that

$$-1 = 1 - 2 \leq 1 - \frac{b}{xy} \leq 1,$$

whence

$$|f(x) - f(y)| = \frac{1}{2}|x - y| \left|\frac{b}{xy} - 1\right| \leq \frac{1}{2}|x - y|.$$

Thus f is a contraction mapping on $[\sqrt{b/2}, \infty)$ with constant $k = 1/2$.

Taking any initial guess $x_0 \geq \sqrt{b/2}$ gives a sequence $(x_n)_{n=0}^\infty$ determined by $x_{n+1} = f(x_n)$ that converges (quite rapidly because $k = 1/2$) to the square root c.

Example (in lieu of 7.1.13). A subset of the Banach space $(C([0,1],\mathbb{R}), \|\cdot\|_\infty)$ is the set

$$D = \{x(t) \in C([0,1],\mathbb{R}) : x(0) = 0\}.$$

This D is a subspace because for $x(t), y(t) \in D$ we have $ax(0) + by(0) = 0$ for all $a, b \in \mathbb{R}$.

This D is also closed because for any convergent sequence $(x_n(t))_{n=1}^\infty \subset D$, the limit function $x(t) \in C([0,1],\mathbb{R})$ satisfies $x(0) = \lim x_n(0) = 0$.

Consider the operator $F : D \to D$ defined by

$$F(x)(t) = \int_0^t f(x(\tau))\ d\tau,\ \ t \in [0,1],$$

where $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant $0 \leq L < 1$, i.e., for all $x, y \in \mathbb{R}$ there holds

$$|f(x) - f(y)| \leq L|x - y|.$$

That $F(x)(t) \in D$ when $x(t) \in D$ follows because the continuity of $f(x(\tau))$ on $[0,1]$ implies the continuity of $F(x)(t) = \int_0^t f(x(\tau))\ d\tau$ on $[0,1]$ (see Theorem 5.10.12(v)), and because $F(x)(0) = 0$.

We show that F is a contraction mapping on D.

For $x(t), y(t) \in D$ we have

$$\begin{aligned} |F(x)(t) - F(y)(t)| &= \left| \int_0^1 \big(f(x(\tau)) - f(y(\tau))\big)\ dt \right| \\ &\leq \int_0^t |f(x(\tau)) - f(y(\tau))|\ dt \\ &\leq \int_0^t L|x(\tau) - y(\tau)|\ dt \\ &\leq \int_0^1 L|x(\tau) - y(\tau)|\ dt \\ &\leq \int_0^1 L\|x - y\|_\infty \\ &= L\|x - y\|_\infty. \end{aligned}$$

This implies that

$$\|F(x) - F(y)\|_\infty = \sup_{t\in[0,1]} |F(x)(t) - F(y)(t)| \leq L\|x - y\|_\infty.$$

Since $0 \leq L < 1$, the operator F is a contraction mapping on D.

By the Contraction Mapping Principle, the operator F has a unique fixed point $\bar{x}(t) \in D$, i.e.,

$$\bar{x}(t) = F(\bar{x})(t) = \int_0^t f(\bar{x}(\tau))\ d\tau,\ \ t \in [0,1].$$

By the Fundamental Theorem of Calculus part (i), the function $t \to \int_0^t f(\bar{x}(\tau))\ d\tau$ is differentiable on $(0,1)$ with derivative

$$\frac{d}{dt}\int_0^t f(\bar{x}(\tau))\ dt = f(\bar{x}(t)).$$

This implies, since $\bar{x}(t) = \int_0^t f(\bar{x}(\tau))\ d\tau$, that $\bar{x}(t)$ is also differentiable on $(0,1)$, i.e.,

$$\bar{x}'(t) = f(\bar{x}(t)),\ \ t \in (0,1).$$

Thus we have the existence of a unique solution $\bar{x}(t)$ of the first-order differential equation $x' = f(x)$ on the interval $(0,1)$ that extends continuously to $[0,1]$ and satisfies the initial condition $x(0) = 0$.

This is not quite the Fundamental Existence and Uniqueness Theorem for Initial Value Problems in Ordinary Differential Equations, but it is close.

Are there any functions $f : \mathbb{R} \to \mathbb{R}$ that are Lipschitz with constant $0 \leq L < 1$?

Yes, there are. The function $f(x) = (1/3)e^{-x^2}$ is Lipschitz with constant $L = 2/3$.

By the Mean Value Theorem applied to f on the interval $[x,y]$ we have

$$|f(x) - f(y)| \leq \sup_{t \in (x,y)} |f'(t)|\,|x-y|.$$

The derivative

$$f'(t) = -\frac{2te^{-t^2}}{3}$$

has limit 0 as $t \to \pm\infty$ and its critical points are the zeros of

$$f''(t) = -\frac{2}{3}\left(e^{-t^2} - 2t^2e^{-t^2}\right) = -\frac{2e^{-t^2}}{3}\left(1 - 2t^2\right)$$

which critical points are $t = \pm\sqrt{1/2}$ at which f' has its maximum and minimum values

$$|f'(\pm\sqrt{1/2})| = \left| \mp \frac{2\sqrt{1/2}\,e^{-1/2}}{3} \right| \leq \frac{2}{3}.$$

Thus f is Lipschitz on $\mathbb{R}$ with constant $L = 2/3$.

Theorem 7.1.14. For a nonempty closed subset D of a Banach space $(X, \|\cdot\|)$, if for $f : D \to D$ there is $n \in \mathbb{N}$ such that f^n is a contraction on D, then there exists a unique fixed point $\bar{x} \in D$ for f.

The proof of this is HW (Exercise 7.6).