Math 346 Lecture #8
7.2 Uniform Contraction Mapping Principle

Definition 7.2.1. For D a nonempty subset of a normed linear space $(X, \|\cdot\|)$ and an arbitrary nonempty set B, a function $f : D \times B \to D$ is called a uniform contraction mapping if there exists $0 \leq \lambda < 1$ such that for all $\mathrm{x}_1, \mathrm{x}_2 \in D$ and for all $b \in B$ there holds
$$\|f(\mathrm{x}_1, b) - f(\mathrm{x}_2, b)\| \leq \lambda \|\mathrm{x}_1 - \mathrm{x}_2\|.$$

Note. A uniform contraction mapping $f : D \times B \to D$ is a contraction mapping for each fixed $b \in B$, i.e., the map $f(\cdot, b) : D \to D$ is a contraction mapping (where we use $\cdot$ in $f(\cdot, b)$ to indicate the input from D). Sometimes the map $f(\cdot, b)$ is denoted by f_b.

The uniform part of a uniform contraction mapping is that there is one constant of contraction λ that applies to $f(\cdot, b) : D \to D$ for all $b \in B$.

Example (simplification of 7.2.3). We saw in the previous lecture, for $x \in (0, \infty)$ and fixed $b \in [1, \infty)$, the function $f : (0, \infty) \to (0, \infty)$ defined by
$$f(x) = \frac{1}{2}\left(x + \frac{b}{x}\right)$$
is a contraction mapping on the closed subset $[\sqrt{b/2}, \infty)$.

In other words, by abuse of notation, we have a function $f : (0, \infty) \times (1, \infty) \to (0, \infty)$ defined by
$$f(x, b) = \frac{1}{2}\left(x + \frac{b}{x}\right).$$
for which f_b is the function above.

We show for $D = [1, 3/2]$ (the book uses $D = [1, 2]$) and $B = [1, 2]$, that $f(D \times B) \subset D$, i.e., the restriction of f to $D \times B$ gives a function $f : D \times B \to D$, and that this restriction is a uniform contraction with constant $\lambda = 1/2$.

To see that $f(D \times B) \subset D$ we find the maximum and minimum values of continuous f on the compact $D \times B$ (which exist by the Extreme Value Theorem).

Since
$$Df(x, b) = \left[\frac{1}{2}\left(1 - \frac{b}{x^2}\right), \frac{1}{2x}\right]$$
there are no critical points of f on the interior of $D \times B$.

So the maximum and minimum values of f occur on the boundary of $D \times B$.

The values of f on the boundary of $D \times B$ are given by four functions
$$f(x, 1) = \frac{1}{2}\left(x + \frac{1}{x}\right) \quad \text{the bottom,} \qquad f(x, 2) = \frac{1}{2}\left(x + \frac{2}{x}\right) \quad \text{the top,}$$
$$f(1, b) = \frac{1 + b}{2} \quad \text{the left,} \qquad f(3/2, b) = \frac{1}{2}\left(\frac{3}{2} + \frac{2b}{3}\right) \quad \text{the right.}$$

The left and the right are increasing functions of b, and so the maxima and minima of these functions occur at the endpoints.

The bottom has no critical point in the interior of $D = [1, 3/2]$ because

$$\frac{\partial f}{\partial x}(x,1) = \frac{1}{2}\left(1 - \frac{1}{x^2}\right) > 0 \text{ when } x \in (1, 3/2).$$

So along the bottom the maximum and minimum occur at the endpoints.

The top has one critical point in the interior of $D = [0, 3/2]$ at $x = \sqrt{2}$ because

$$\frac{\partial f}{\partial x}(x,2) = \frac{1}{2}\left(1 - \frac{2}{x^2}\right).$$

Thus there are five points on the boundary of $D \times B$ where the maximum and minimum of f can occur; these are the four corners and the point $(\sqrt{2}, 2)$.

The values of f at these fives points are

$$f(1,1) = 1, \quad f(3/2,1) = \frac{13}{12}, \quad f(1,2) = \frac{3}{2}, \quad f(\sqrt{2},2) = \sqrt{2}, \quad f(3/2,2) = \frac{17}{12}.$$

The maximum and minimum of f restricted to $D \times B$ are $3/2$ and 1.

Thus the image of f restricted to $D \times B$ is precisely D.

Next we show that $f : D \times B \to D$ is a uniform contraction with constant $\lambda = 1/2$.

We showed in the previous lecture that the function f satisfies

$$|f(x,b) - f(y,b)| = \frac{1}{2}|x - y| \left|1 - \frac{b}{xy}\right|.$$

The inequalities

$$\left|1 - \frac{b}{xy}\right| \leq 1 \text{ and } 2xy \geq b \geq 0$$

are equivalent because

$$-1 \leq 1 - \frac{b}{xy} \leq 1 \Leftrightarrow -2 \leq -\frac{b}{xy} \leq 0 \Leftrightarrow 2 \geq \frac{b}{xy} \geq 0 \Leftrightarrow 2xy \geq b \geq 0.$$

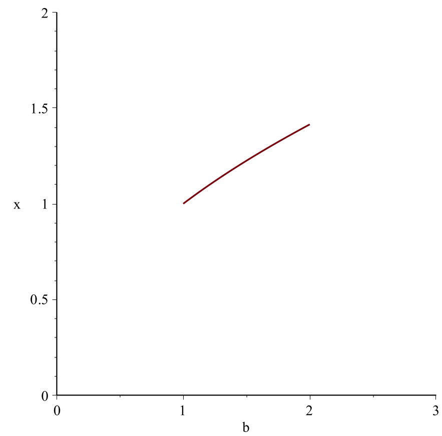
The inequality $|1 - b/xy| \leq 1$ holds on $D \times B$ because $2xy \geq b \geq 0$ when $x, y \in [1, 3/2]$ and $b \in [1, 2]$, i.e., the smallest $2xy$ can be is 2 and the biggest b can be is 2.

Thus the function $f : D \times B \to D$ is a uniform contraction mapping with $\lambda = 1/2$.

For each fixed $b \in [1, 2]$ there is a unique fixed point f_b.

This defines a function $g : B \to D$ that takes b to the unique fixed point of f_b.

See the following graph of g that shows for each b the unique fixed point of f_b.



What can we say about this function g? It is continuous? Differentiable?

We can answer these because we know that $g(b) = \sqrt{b}$ which is a continuous and differentiable.

7.2.1 The Uniform Contraction Mapping Principle

Remark 7.2.5. To define the derivative of a function f whose domain U is not open (i.e., the closure of an open set), we consider for those x not in the interior of U only those $\mathrm{h} \in X$ for which $\mathrm{x}+\mathrm{h}$ belongs to U in the limit defining the derivative.

Theorem 7.2.4 (Uniform Contraction Mapping Principle). For Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, open U in X and open V in Y, if a function $f : \overline{U} \times V \to \overline{U}$ is uniform contraction mapping with constant $0 \leq \lambda < 1$ and $f \in C^k(\overline{U} \times V, \overline{U})$ for some $k \in \mathbb{N}$, then the function $g : V \to \overline{U}$ that sends each $\mathrm{y} \in V$ to the unique fixed point of the contraction $f(\cdot, \mathrm{y})$ belongs to $C^k(V, \overline{U})$.

We give an outline of the proof.

First one proves the continuity g as a consequence of the continuity of f, i.e., the case $k = 0$, and the differentiability of g when f is C^1, i.e., the case $k = 1$. Then one inducts on k.

Lemma 7.2.6 gives the continuity of g under slightly weaker conditions than required in the theorem: if f is continuous at each $(g(\mathrm{y}), \mathrm{y}) \in \overline{U} \times V$, then g is continuous at each $\mathrm{y} \in V$.

By the continuity of f at $(g(\mathrm{y}), \mathrm{y})$, for $\epsilon > 0$ there exists $\delta > 0$ such that for all $\mathrm{h} \in B(\mathrm{y}, \delta) \subset V$ there holds

$$\|f(g(\mathrm{y}), \mathrm{y}+\mathrm{h}) - f(g(\mathrm{y}), \mathrm{y})\| \leq (1-\lambda)\epsilon.$$

Using that $g(\mathrm{y})$ is a fixed point of f_{y} and that $f_{\mathrm{y}+\mathrm{h}}$ is a contraction with constant λ, we have

$$\begin{aligned}
&\|g(\mathrm{y}+\mathrm{h}) - g(\mathrm{y})\| \\
&\quad = \|f(g(\mathrm{y}+\mathrm{h}), \mathrm{y}+\mathrm{h}) - f(g(\mathrm{y}), \mathrm{y})\| \\
&\quad \leq \|f(g(\mathrm{y}+\mathrm{h}), \mathrm{y}+\mathrm{h}) - f(g(\mathrm{y}), \mathrm{y}+\mathrm{h})\| + \|f(g(\mathrm{y}), \mathrm{y}+\mathrm{h}) - f(g(\mathrm{y}), \mathrm{y})\| \\
&\quad \leq \lambda\|g(\mathrm{y}+\mathrm{h}) - g(\mathrm{y})\| + (1-\lambda)\epsilon.
\end{aligned}$$

This implies by combining the common terms that

$$(1-\lambda)\|g(\mathrm{y}+\mathrm{h})-g(\mathrm{y})\| \leq (1-\lambda)\epsilon.$$

Since $0 \leq \lambda < 1$, we obtain

$$\|g(\mathrm{y}+\mathrm{h})-g(\mathrm{y})\| \leq \epsilon.$$

This shows that g is continuous at y and completes the case $k=0$.

In the case $k=1$, i.e., $f \in C^1(\overline{U}\times V, \overline{U})$, one has to get a good guess for the derivative of g.

Lemma 7.2.7 gives a way to find a good guess for the derivative of g under slightly weaker conditions than required in the theorem: if f is C^1 at each $(g(\mathrm{y}),\mathrm{y}) \in \overline{U}\times V$, then the function $\phi : \mathscr{B}(Y,X)\times V \to \mathscr{B}(Y,X)$ defined by

$$\phi(A,y) = D_1 f(g(\mathrm{y}),\mathrm{y})A + D_2 f(g(\mathrm{y}),\mathrm{y})$$

is a uniform contraction.

Since $f(\cdot,\mathrm{y})$ is a uniform contraction on $\overline{U}$ it is locally Lipschitz at $g(\mathrm{y})$ with constant λ; we have by Proposition 6.3.10 (simplified version) that in the induced norm there holds

$$\|D_1 f(g(\mathrm{y}),\mathrm{y})\| \leq \lambda.$$

Thus

$$\begin{aligned}
&\|\phi(A_2,\mathrm{y})-\phi(A_2,\mathrm{y})\| \\
&\quad = \|D_1 f(g(\mathrm{y}),\mathrm{y})A_2 + D_2 f(g(\mathrm{y}),\mathrm{y}) - D_1 f(g(\mathrm{y}),\mathrm{y})A_1 - D_2 f(g(\mathrm{y}),\mathrm{y})\| \\
&\quad = \|D_1 f(g(\mathrm{y}),\mathrm{y})(A_2-A_1)\| \\
&\quad \leq \|D_1 f(g(\mathrm{y}),\mathrm{y})\|\,\|A_2-A_1\| \\
&\quad \leq \lambda\|A_2-A_1\|.
\end{aligned}$$

Remark 7.2.8. Lemma 7.2.7 implies that for each $\mathrm{y} \in V$ there exists a unique fixed point $Z(\mathrm{y}) \in \mathscr{B}(Y,X)$ of ϕ, i.e.,

$$Z(\mathrm{y}) = D_1 f(g(\mathrm{y}),\mathrm{y})Z(\mathrm{y}) + D_2 f(g(\mathrm{y}),\mathrm{y}).$$

The function $Z : V \to \mathscr{B}(Y,X)$ is continuous by an argument similar to that of the proof of Lemma 7.2.6 because Df is continuous at each $(g(\mathrm{y}),\mathrm{y})$.

Lemma 7.2.9 proves that the good guess $Z(\mathrm{y})$ for the derivative of g at y satisfies for $\epsilon > 0$ there exists $\delta > 0$ such that for all $\mathrm{h} \in B(\mathrm{y},\delta) \subset V$ there holds

$$\|g(\mathrm{y}+\mathrm{h})-g(\mathrm{y})-Z(\mathrm{y})\mathrm{h}\| \leq \epsilon\|\mathrm{h}\|.$$

The proof of Lemma 7.2.9 is long and tedious. It is a starred subsection in the book.

Example. Recall from the previous lecture that

$$D = \{x(t) \in C([0,1],\mathbb{R}) : x(0) = 0\}$$

is a closed subspace of $(C([0,1],\mathbb{R}), \|\cdot\|\infty)$.

For an open interval $I \subset \mathbb{R}$, a constant $0 \leq L < 1$, and a function $f : \mathbb{R} \times I \to \mathbb{R}$ that satisfies

$$|f(x,b) - f(y,b)| \leq L|x-y| \ \text{ for all } x, y \in \mathbb{R}, b \in I,$$

define the operator $F : D \times I \to D$ by

$$F(x,b)(t) = \int_0^t f(x(\tau), b)\ d\tau,\ \ t \in [0,1].$$

The operator F is a uniform contraction because

$$\begin{aligned}|F(x,b)(t) - F(y,b)(t)| &\leq \int_0^t |f(x(\tau),b) - f(y(\tau),b)|\ d\tau \\ &\leq \int_0^t L|x(\tau) - y(\tau)|\ d\tau \leq L\|x-y\|_\infty.\end{aligned}$$

implies that

$$\|F(x,b) - F(y,b)\|_\infty \leq L\|x-y\|_\infty.$$

Thus for each fixed $b \in I$, there exists a unique fixed point $x_b \in D$ of F, i.e.,

$$x_b(t) = \int_0^t f(x_b(\tau), b)\ d\tau,\ \ t \in [0,1].$$

Is the map $b \to g_b$ continuous?

The uniform contraction mapping condition

$$|f(x,b) - f(y,b)| \leq L|x-y| \ \text{ for all } x, y \in \mathbb{R}, b \in I$$

on f is not enough to imply that $F \in C^0(D \times I, D)$.

However the condition of $f : \mathbb{R} \times I \to \mathbb{R}$ being uniformly continuous is enough to imply $F \in C^0(D \times I, D)$.

For the 1-norm $\|(x,b) - (y,c)\|_1 = |x-y| + |b-c|$ on $\mathbb{R} \times \mathbb{R}$, uniform continuity of f on $\mathbb{R} \times I$ is: for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $\|(x,b) - (y,c)\|_1 < \delta$ there holds

$$|f(x,b) - f(y,c)| < \epsilon.$$

[Recall that all norms on finite dimensional vector spaces are topologically equivalent.]

Then at a fixed $(x(t), b) \in D \times I$ we have for all $y(t) \in B(x(t), \delta/2) \cap D$ and $|b-c| < \delta/2$ that for each $t \in [0,1]$ there holds

$$\|(x(t),b) - (y(t),c)\|_1 = |x(t) - y(t)| + |b-c| \leq \|x-y\|_\infty + |b-c| < \delta,$$

and so

$$|f(x(t),b) - f(y(t),c)| < \epsilon \text{ for all } t \in [0,1].$$

This implies that

$$\begin{aligned}
\sup_{t\in[0,1]} |F(x,b)(t) - F(y,c)(t)| &\leq \sup_{t\in[0,1]} \int_0^t |f(x(\tau),b) - f(y(\tau),c)| \; d\tau \\
&\leq \sup_{t\in[0,1]} \int_0^t \epsilon \; d\tau \\
&\leq \sup_{t\in[0,1]} \int_0^1 \epsilon \; d\tau \\
&= \epsilon.
\end{aligned}$$

This shows that uniform continuity of $f : \mathbb{R} \times I \to \mathbb{R}$ implies that $F \in C^0(D \times I, D)$.

By the Uniform Contraction Mapping Principle the function $b \to x_b$ is continuous.

Note. The condition $f \in C^k(\mathbb{R} \times I, \mathbb{R})$ does imply that $F \in C^k(D \times I, D)$, which implies by the Uniform Contraction Mapping Principle that $b \to g_b$ is C^k.

An example of a C^∞ function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$f(x,b) = \frac{e^{-(x-b)^2}}{3}.$$

This function is uniformly continuous and is a uniform contraction, i.e.,

$$|f(x,b) - f(y,b)| \leq \frac{2}{3}|x-y|.$$