Math 346 Lecture #12
8.1 Multivariable Integration

We extend the construction of the regulated integral for functions of a single variable to functions of several variables by defining the integral for step functions and then applying the continuous linear extension theorem.

8.1.1 Multivariable Step Functions

Definition 8.1.1. For $\mathrm{a} = (a_1, \ldots, a_n)$ and $\mathrm{b} = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$ with $a_i \leq b_i$ for all $i = 1, \ldots, n$, the closed n-interval $[\mathrm{a}, \mathrm{b}]$ is defined to be

$$[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n.$$

The closed n-interval $[\mathrm{a}, \mathrm{b}]$ is a compact box or parallelepiped in $\mathbb{R}^n$.

Definition 8.1.2. A subdivision $\mathscr{P}$ of a closed n-interval $[\mathrm{a}, \mathrm{b}]$ consists, for each $i = 1, \ldots, n$, of a subdivsion

$$P_i = \{t_i^{(0)} = a_i < t_i^{(1)} < \cdots < t_i^{(k_i-1)} < t_i^{(k_i)} = b_i\}$$

of the closed interval $[a_i, b_i]$ for some $k_i \in \mathbb{N}$.

Definition 8.1.3. A subdivision of an n-interval $[\mathrm{a}, \mathrm{b}]$ gives a decomposition of $[\mathrm{a}, \mathrm{b}]$ into a disjoint union of an partially open subinterval and closed subinterval as follows. Each $t_i^{(j)}$ with $0 < j < k_i$ defines a hyperplane

$$H_i^{(j)} = \{\mathrm{x} \in \mathbb{R}^n : x_i = t_i^{(j)}\}$$

in $\mathbb{R}^n$ which divides $[\mathrm{a}, \mathrm{b}]$ into two regions; a partially open subinterval

$$[a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_i, t_i^{(j)}) \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n],$$

and a closed subinterval

$$[a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [t_i^{(j)}, b_i] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_n, b_n].$$

(Sketch the picture in $\mathbb{R}^2$.)

Repeating this process for each resulting subinterval (either partially open or closed) and for each hyperplane, we get a decomposition of $[\mathrm{a}, \mathrm{b}]$ into a pairwise disjoint union of subintervals.

We account for each these subintervals R_I through an n-tuple $I = (i_1, \ldots, i_n)$ where $i_j \in \{1, \ldots, k_j\}$, which is to say that each subinterval is associated with the corner whose coordinates are

$$(t_1^{i_1}, \ldots, t_n^{i_n}).$$

(In $\mathbb{R}^2$, this point is the top right corner.)

This association gives a bijection from

$$\{1, \ldots, k_1\} \times \cdots \times \{1, \ldots, k_n\}$$

to $\mathscr{P}$ by which we identify $(i_1, \ldots, i_n)$ with $(t_1^{i_1}, \ldots, t_n^{i_n})$.

All except one of the subintervals R_I are partially open, i.e., have a least one face missing; the exception is the closed subinterval R_I for $I = (k_1, \ldots, k_n)$.

We thus have a pairwise disjoint union of $[\mathrm{a}, \mathrm{b}]$ into subintervals:

$$[\mathrm{a}, \mathrm{b}] = \bigcup_{I \in \mathscr{P}} R_I.$$

We assume throughout the remainder of this lecture that $(X, \|\cdot\|)$ is a Banach space over $\mathbb{R}$.

Definition 8.1.4. For a subset E of $\mathbb{R}^n$ the indicator or characteristic function on E is

$$\chi_E(\mathrm{x}) = \begin{cases} 1 & \text{if } \mathrm{x} \in E, \\ 0 & \text{if } \mathrm{x} \notin E. \end{cases}$$

A function $s : [\mathrm{a}, \mathrm{b}] \to X$ is a step function if there exists a subdivision $\mathscr{P}$ of $[\mathrm{a}, \mathrm{b}]$ and elements $\mathrm{x}_I \in X$ for each $I \in \mathscr{P}$ such that

$$s(\mathrm{x}) = \sum_{I \in \mathscr{P}} \mathrm{x}_I \chi_{R_I}(\mathrm{x}).$$

More generally we consider a function $s : E \to X$ to be a step function if it is zero outside some interval $[\mathrm{a}, \mathrm{b}]$ and the restriction of s to $[\mathrm{a}, \mathrm{b}]$ is a step function on $[\mathrm{a}, \mathrm{b}]$.

Let $S([\mathrm{a}, \mathrm{b}], X)$ denote the collection of all step functions $s : [\mathrm{a}, \mathrm{b}] \to X$. This collection is nonempty because the zero function is a step function.

Proposition 8.1.5. The collection $S([\mathrm{a}, \mathrm{b}], X)$ is a subspace of the Banach space, $(L^\infty([\mathrm{a}, \mathrm{b}], X), \|\cdot\|_\infty)$, of bounded functions from $[\mathrm{a}, \mathrm{b}]$ to X.

The proof of this is nearly identical to the single variable counterpart (see Proposition 5.10.3).

8.1.2 Multivariable, Banach-Valued Integration

Every n-interval has a naturally defined n-dimensional volume or "measure."

Definition 8.1.6. For each $j \in \{1, \ldots, n\}$ let A_j be an interval of one of the forms (a_j, b_j), $[a_j, b_j)$, $(a_j, b_j]$, or $[a_j, b_j]$ for $a_j \leq b_j$ and $a_j, b_j \in \mathbb{R}$. We define the measure of $R = A_1 \times \cdots \times A_n$ to be

$$\lambda(R) = \prod_{j=1}^{n} (b_j - a_j).$$

Each R with nonempty interior will have a positive measure, while any R with some $a_j = b_j$ will have zero measure.

Definition 8.1.7. The integral of a step function

$$s(\mathrm{x}) = \sum_{I \in \mathscr{P}} \mathrm{x}_I \chi_{R_I}(\mathrm{x})$$

in $S([\mathrm{a},\mathrm{b}],X)$ is

$$\mathscr{I}(s)=\int_{[\mathrm{a},\mathrm{b}]} s=\sum_{I\in\mathscr{P}} \mathrm{x}_I\lambda(R_I),$$

a finite linear combination in the Banach space X.

Proposition 8.1.8. For any n-interval $[\mathrm{a},\mathrm{b}]\subset\mathbb{R}^n$ (which n-interval is compact by definition), the integral operator $\mathscr{I}:S([\mathrm{a},\mathrm{b}],X)\to X$ is a bounded linear transformation where

$$\|\mathscr{I}\|=\lambda([\mathrm{a},\mathrm{b}]).$$

The proof of this is HW (Exercise 8.2). [Notice that there is a typo in the book: $\lambda([\mathrm{b}-\mathrm{a}])$ should be $\lambda([\mathrm{a},\mathrm{b}])$. This typo appears again in Theorem 8.1.9.]

Note. Recall that we showed in the lecture note for Section 5.7 that the closure of a subspace is a subspace (a result not mentioned nor proved in the book).

Theorem 8.1.9 (Multivariable, Banach-Valued Integral). The bounded linear transformation $\mathscr{I}:S([\mathrm{a},\mathrm{b}],X)\to X$ extends uniquely to a bounded linear transformation $\overline{\mathscr{I}}:\overline{S([\mathrm{a},\mathrm{b}],X)}\to X$ such that

$$\|\overline{\mathscr{I}}\|=\lambda([\mathrm{a},\mathrm{b}]).$$

Moreover we have

$$C([\mathrm{a},\mathrm{b}],X)\subset\overline{S([\mathrm{a},\mathrm{b}],X)}\subset L^\infty([\mathrm{a},\mathrm{b}],X).$$

The proof of this is HW (Exercise 8.3).

Definition 8.1.10. For any compact n-interval $[\mathrm{a},\mathrm{b}]$ we denote the set $\overline{S([\mathrm{a},\mathrm{b}],X)}$ by $\mathscr{R}([\mathrm{a},\mathrm{b}],X)$ which means the closed subspace of regulated-integrable functions.

For $f\in\mathscr{R}([\mathrm{a},\mathrm{b}],X)$ we call the the bounded linear transformation $\overline{\mathscr{I}}$ the integral of f and write

$$\int_{[\mathrm{a},\mathrm{b}]} f=\overline{\mathscr{I}}(f).$$

Proposition 8.1.11. For $f,g\in\mathscr{R}([\mathrm{a},\mathrm{b}],X)$, the following hold.

(i) $\left\|\int_{[\mathrm{a},\mathrm{b}]} f\right\|\le\lambda([\mathrm{a},\mathrm{b}])\sup_{t\in[\mathrm{a},\mathrm{b}]}\|f(t)\|.$

(ii) For a sequence $(f_n)_{n=1}^\infty$ in $\mathscr{R}([\mathrm{a},\mathrm{b}],X)$ converging uniformly to f there holds

$$\lim_{n\to\infty}\int_{[\mathrm{a},\mathrm{b}]} f_n=\int_{[\mathrm{a},\mathrm{b}]}\lim_{n\to\infty} f_n=\int_{[\mathrm{a},\mathrm{b}]} f.$$

(iii) With $\|f\|$ denoting the function $t\to\|f(t)\|$ from $[\mathrm{a},\mathrm{b}]$ to $\mathbb{R}$, there holds

$$\left\|\int_{[\mathrm{a},\mathrm{b}]} f\right\|\le\int_{[\mathrm{a},\mathrm{b}]}\|f\|.$$

(iv) If $\|f(t)\| \le \|g(t)\|$ for all $t \in$ [a, b], then

$$\int_{[\mathrm{a,b}]} \|f\| \le \int_{[\mathrm{a,b}]} \|g\|.$$

Proof. Parts (i) and (ii) are HW (Exercise 8.5).

(iii) For a step function

$$s(t) = \sum_{I\in\mathscr{P}} \mathrm{x}_I \chi_{R_I}(t) \in \mathscr{R}([\mathrm{a,b}], X)$$

we have by the pairwise disjointness of the R_I that

$$\|s(t)\| = \sum_{I\in\mathscr{P}} \|\mathrm{x}_I\| \chi_{R_I}(t) \in \mathscr{R}([\mathrm{a,b}], \mathbb{R}).$$

Since s is a finite sum, we have by the triangle inequality that

$$\left\| \int_{[\mathrm{a,b}]} s \right\| = \left\| \sum_{I\in\mathscr{P}} \mathrm{x}_I \lambda(R_I) \right\| \le \sum_{I\in\mathscr{P}} \|x_I\| \lambda(R_I) = \int_{[\mathrm{a,b}]} \|s\|.$$

For any $f \in \mathscr{R}([\mathrm{a,b}], X)$ there is a sequence of step functions $(s_n)_{n=1}^\infty$ such that $s_n \to f$ uniformly on [a, b].

This implies by the continuity of the norm and part (ii) that

$$\begin{aligned}
\left\| \int_{[\mathrm{a,b}]} f \right\| &= \left\| \int_{[\mathrm{a,b}]} \lim_{n\to\infty} s_n \right\| \\
&= \lim_{n\to\infty} \left\| \int_{[\mathrm{a,b}]} s_n \right\| \\
&\le \lim_{n\to\infty} \int_{[\mathrm{a,b}]} \|s_n\| \\
&= \int_{[\mathrm{a,b}]} \left\| \lim_{n\to\infty} s_n \right\| \\
&= \int_{[\mathrm{a,b}]} \|f\| .
\end{aligned}$$

(iv) Suppose $h \in \mathscr{R}([\mathrm{a,b}], \mathbb{R})$ satisfies $h(t) \ge 0$ for all $t \in$ [a, b].

There is a sequence of step functions $(s_n)_{n=1}^\infty$ that converges uniformly to f on [a, b]: for $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \ge N$ there holds

$$\|s_n - h\|_\infty \le \frac{\epsilon}{\lambda([\mathrm{a,b}])}.$$

Since $h(t) \ge 0$ for all $t \in$ [a, b], the uniform convergence implies for all $n \ge N$ that

$$\frac{\epsilon}{\lambda([\mathrm{a,b}])} \ge |h(t) - s_n(t)| \ge h(t) - s_n(t) \ge -s_n(t) \text{ for all } t \in [\mathrm{a,b}].$$

This implies for all $n \geq N$ that

$$s_n(t) \geq -\frac{\epsilon}{\lambda([\mathrm{a},\mathrm{b}])} \text{ for all } t \in [\mathrm{a},\mathrm{b}].$$

Consequently, as

$$s_n(t) = \sum_{I \in \mathscr{P}} x_I \chi_{R_I}(t)$$

for $x_I \in \mathbb{R}$, it follows for all $I \in \mathscr{P}$ that

$$x_I \geq -\frac{\epsilon}{\lambda([\mathrm{a},\mathrm{b}])}.$$

Hence for each $n \geq N$ we have

$$\begin{aligned}\int_{[\mathrm{a},\mathrm{b}]} s_n = \sum_{I \in \mathscr{P}} x_I \lambda(R_I) &\geq -\sum_{I \in \mathscr{P}} \frac{\epsilon \lambda(R_I)}{\lambda([\mathrm{a},\mathrm{b}])} \\ &= -\frac{\epsilon}{\lambda([\mathrm{a},\mathrm{b}])} \sum_{I \in \mathscr{P}} \lambda(R_I) \\ &= -\frac{\epsilon}{\lambda([\mathrm{a},\mathrm{b}])} \lambda([\mathrm{a},\mathrm{b}]) \\ &= -\epsilon.\end{aligned}$$

By part (ii) we have for all $n \geq N$ that

$$\int_{[\mathrm{a},\mathrm{b}]} h = \lim_{n \to \infty} \int_{[\mathrm{a},\mathrm{b}]} s_n \geq -\epsilon.$$

Since this holds for any $\epsilon > 0$ we conclude that

$$\int_{[\mathrm{a},\mathrm{b}]} h \geq 0.$$

By setting $h(t) = \|f(t)\| - \|g(t)\|$ we obtain the result. □

Remark 8.1.12. The Riemann construction of the integral defines a bounded linear transformation on $\mathscr{R}([\mathrm{a},\mathrm{b}], X)$ that agrees with the regulated integral on step functions. Hence by the uniqueness part of the Continuous Linear Extension Theorem, the Riemann integral and the regulated integral agree on $\mathscr{R}([\mathrm{a},\mathrm{b}], X)$.

8.1.3 Integration over subsets of [a, b]

To integrate functions defined on bounded subsets E of $\mathbb{R}^n$ other than closed n-intervals, we extend the functions by zero outside of E.

Definition 8.1.13. For any function $f : E \to X$, the extension of f by zero is the function

$$f\chi_E(z) = \begin{cases} f(z) & \text{if } z \in E, \\ 0 & \text{if } z \notin E. \end{cases}$$

Since E is bounded in $\mathbb{R}^n$, its closure is compact, and there is a compact n-interval $[\mathrm{a}, \mathrm{b}]$ that contains E.

We could then define the integral of f to be

$$\int_E f = \int_{[\mathrm{a},\mathrm{b}]} f \chi_E.$$

An immediate problem with doing this is that we don't know beforehand if $f\chi_E$ belongs to $\mathscr{R}([\mathrm{a}, \mathrm{b}], X)$.

It is even possible that the indicator function χ_E may not be integrable.

Unexample 8.1.14. For an compact 1-interval $[a, b]$ with $a < b$, the singleton set $E = \{p\}$ for $p \in [a, b)$ has χ_E not integrable.

This follows because every step function $s : [a, b] \to \mathbb{R}$ is right continuous, meaning for every $t_0 \in [a, b)$ there holds

$$\lim_{t \to t_0^+} s(t) = s(t_0).$$

By Exercise 8.4 (a HW problem) the uniform limit of right-continuous functions is a right-continuous function.

But the indicator function χ_E is not right-continuous at $t_0 = p$, and therefore is not integrable.

Overcoming this and other deficiencies of the regulated integral is discussed in the next section.