Math 346 Lecture #13
8.2 Overview of Daniell-Lebesgue Integration

8.2.1 The Problem

The regulated integral is a bounded linear transformation defined on the closed subspace $\mathscr{R}([\mathrm{a,b}],X) = \overline{S([\mathrm{a,b}],X)}$ of the Banach space $(L^\infty([\mathrm{a,b}],X), \|\cdot\|_\infty)$.

The closed-ness of $\mathscr{R}([\mathrm{a,b}],X)$ means that $\mathscr{R}([\mathrm{a,b}],X)$ is also complete: any Cauchy sequence $(f_n)_{n=1}^\infty$ in $\mathscr{R}([\mathrm{a,b}],X)$ converges uniformly to an $f \in \mathscr{R}([\mathrm{a,b}],X)$ and we have the really useful commutativity of limit and integral,

$$\lim_{n\to\infty} \int_{[\mathrm{a,b}]} f_n = \int_{[\mathrm{a,b}]} f.$$

Although $\mathscr{R}([\mathrm{a,b}],X)$ contains $C([\mathrm{a,b}],X)$, it fails to include functions that should be integrable (say the functions $f : [a,b] \to \mathbb{R}$, for $[a,b] \subset \mathbb{R}$, that are monotone but not right continuous).

The problem is that we cannot extend again the regulated integral in the ∞-norm to a larger closed subspace of $L^\infty([\mathrm{a,b}],X)$ because $\mathscr{R}([\mathrm{a,b}],X)$ is already complete in the ∞-norm.

The idea is to find another norm for which $\mathscr{R}([\mathrm{a,b}],X)$ is not complete and apply the continuous linear extension theorem in this norm to extend integration.

Definition 8.2.1. We define a function $\|\cdot\|_1 : \mathscr{R}([\mathrm{a,b}],X) \to \mathbb{R}$ by

$$\|f\|_1 = \int_{[\mathrm{a,b}]} \|f\|.$$

Proposition 8.2.2. The function $\|\cdot\|_1$ is a norm on $\mathscr{R}([\mathrm{a,b}],X)$.

The proof of this is HW (Exercise 8.6).

We call $\|f\|_1$ the L^1-norm of $f \in \mathscr{R}([\mathrm{a,b}],X)$.

Nota Bene 8.2.3. We will focus attention on the case when $X = \mathbb{R}$, although most of the ideas work just fine for an arbitrary Banach space X.

When X is a finite-dimensional Banach space over a field $\mathbb{F}$ being either $\mathbb{R}$ or $\mathbb{C}$, it suffices to describe integration with values in $\mathbb{R}$ and extend this to X.

When $X = \mathbb{C}$ the integral of $f \in \mathscr{R}([\mathrm{a,b}],\mathbb{C})$ is realized as two real-valued integrals because for $f = u + iv$ we set

$$\int_{[\mathrm{a,b}]} f = \int_{[\mathrm{a,b}]} u + i\int_{[\mathrm{a,b}]} v.$$

When $X = \mathbb{F}^n$ the integral of $f = (f_1,\ldots,f_n) \in \mathscr{R}([\mathrm{a,b}],X)$ is realized as

$$\int_{[\mathrm{a,b}]} f = \left(\int_{[\mathrm{a,b}]} f_1, \ldots, \int_{[\mathrm{a,b}]} f\right).$$

Although we do not yet know what the closure of $S([\mathrm{a},\mathrm{b}],\mathbb{R})$ is with respect to the L^1-norm, or even if its exists, we denote this closure by $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$.

We will describe the how to construct a vector space $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ that contains both $S([\mathrm{a},\mathrm{b}],\mathbb{R})$ and $\mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R})$.

We will also outline how to define the integral and the L^1-norm on $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ so that

(a) the new integral agrees with the regulated integral on $\mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R})$,

(b) the new integral is a bounded linear transformation with respect to the L^1-norm, and

(c) the normed linear space $(L^1([\mathrm{a},\mathrm{b}],\mathbb{R}),\|\cdot\|_1)$ is a Banach space.

The full proofs of all of these are found in Chapter 9 (which we are skipping), but hopefully you will get enough of the ideas involved to see what is happening without the long and tedious proofs.

8.2.2 Sketch of Daniell-Lebesgue Integration

We first sketch the construction of the completion $\hat{S}$ of any incomplete normed linear space $(S,\|\cdot\|)$, and then apply this to the normed linear space $(S([\mathrm{a},\mathrm{b}],\mathbb{R}),\|\cdot\|_1)$.

Construction of a Completion. We start with the collection of all the Cauchy sequences in S:

$$S' = \{(s_n)_{n=1}^\infty \subset S : (s_n)_{n=1}^\infty \text{ is Cauchy in } S\}.$$

The collection S' is a vector space with vector addition and scalar multiplication defined by

$$\alpha(s_n)_{n=1}^\infty + \beta(t_n)_{n=1}^\infty = (\alpha s_n + \beta t_n)_{n=1}^\infty,$$

that is, for scalars α and β the sequence $(\alpha s_n + \beta t_n)_{n=1}^\infty$ is Cauchy because for $m,n\in\mathbb{N}$ we have

$$\|\alpha s_m + \beta t_m - \alpha s_n - \beta t_n\|_1 \le \alpha\|s_n - s_m\|_1 + \beta\|t_m - t_n\|_1$$

which will be small for sufficiently large m and n since $(s_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$ are Cauchy.

We "define" a real-valued function $\|\cdot\|'$ on S' by

$$\|(s_n)_{n=1}^\infty\|' = \lim_{n\to\infty}\|s_n\|.$$

That $\|\cdot\|'$ is well-defined (i.e., the limit exists) follows from the reverse triangle inequality,

$$\big|\,\|s_m\| - \|s_n\|\,\big| \le \|s_m - s_n\|,$$

and Cauchy-ness of $(s_n)_{n=1}^\infty$, so that $(\|s_n\|)_{n=1}^\infty$ is Cauchy in $\mathbb{R}$.

Unfortunately the function $\|\cdot\|'$ is not a norm because $\|(s_n)_{n=1}^\infty\|' = 0$ does not imply that $(s_n)_{n=1}^\infty$ is the zero element of S', i.e., the sequence where $s_n = 0$ for all n.

[The set $\mathbb{Q}$ with the norm $|\cdot|$ is an incomplete normed linear space, where the Cauchy sequence $(1/n)_{n=1}^\infty$ has limit zero but is not the zero element of Q'.]

However, the set

$$K = \{(s_n)_{n=1}^\infty \in S' : \|(s_n)_{n=1}^\infty\|' = 0\}$$

is a vector subspace of S'.

The function $\|\cdot\|'$ induces a norm $\|\cdot\|_{\hat{S}}$ on the quotient vector space

$$\hat{S} = S'/K.$$

The elements of $\hat{S}$ are equivalent classes of Cauchy sequences where two Cauchy sequences $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are equivalent if and only if the Cauchy sequence $(s_n - t_n)_{n=1}^{\infty}$ has $\|(s_n - t_n)_{n=1}^{\infty}\|' = 0$, i.e., $\|s_n - t_n\| \to 0$.

The pair $(\hat{S}, \|\cdot\|_{\hat{S}})$ is a Banach space.

The function $S \to S'$ given by $s \to (s)_{n=1}^{\infty}$ (the constant sequence with each entry being s) is a linear injection whose range is dense in $\hat{S}$ with $\|s\| = \|(s)_{n=1}^{\infty}\|_{S'}$ for all $s \in S$.

We may thus think of S as a dense subspace of $\hat{S}$.

Application to $S([\mathrm{a}, \mathrm{b}], \mathbb{R})$. We apply this construction to the incomplete normed linear space $(S([\mathrm{a}, \mathrm{b}], \mathbb{R}), \|\cdot\|_1)$.

We denote the completion $\hat{S}([\mathrm{a}, \mathrm{b}], \mathbb{R})$ here by $L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$.

Two issues arise when trying to define the integral of an element of $L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$.

First an element of $L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$ is an equivalence class of Cauchy sequences of step functions; second, two equivalent Cauchy sequences can converge pointwise to different functions, as illustrated next.

Unexample 8.2.5. The sequence $(s_n)_{n=1}^{\infty}$ in $S([0, 1], \mathbb{R}$ defined by $s_n = \chi_{[0,2^{-n})}$ is Cauchy with limit function $s = \chi_{\{0\}}$.

The sequence $(t_n)_{n=1}^{\infty}$ in $S([0, 1], \mathbb{R})$ defined by $t_n = 0$ for all n, is Cauchy with pointwise limit function $t = 0$.

The Cauchy sequences $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are equivalent because

$$\|s_n - t_n\|_1 = \int_{[0,1]} \chi_{[0,2^{-n})} = 2^{-n} \to 0.$$

But the pointwise limit functions of $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ are not the same.

Resolution of Two Issues. To resolve the first issue, one proves that in each equivalence class of Cauchy sequences there exists a Cauchy sequence that converges pointwise to a function.

This gives a candidate for a function to integrate.

To resolve the second issue – strictly speaking there is no way to resolve this.

Instead we accept that there can be two different pointwise limit functions associated to a given equivalence class of Cauchy sequences of step functions.

Luckily any two pointwise limit functions arising from the same equivalence class will only differ on a set of measure zero (like the middle-thirds Cantor set in $\mathbb{R}$).

We will develop the notion of sets of measure zero more in the next section.

The relation of two functions differing on a set of measure zero is an equivalence relation.

Definition. Two functions $f, g : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ are equal almost everywhere, denoted by $f = g$ a.e., if

$$\{\mathrm{x} \in [\mathrm{a}, \mathrm{b}] : f(\mathrm{x}) - g(\mathrm{x}) \neq 0\}$$

is a set of measure zero.

We now have associated to each equivalence class of Cauchy sequences of step functions a unique equivalence class of pointwise limits functions that differ only on sets of measure zero.

As we shall see, sets of measure zero make no difference to the integral, and thus we can associate a unique integral value (if the integral exists) to each equivalence class of Cauchy sequences of step functions.

That the integral exists for each "element" of $L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$ follows from the Continuous Linear Extension Theorem because the integral on step functions is a bounded linear transformation with respect to the L^1-norm and because $S([\mathrm{a}, \mathrm{b}], \mathbb{R})$ is a dense subspace of $L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$.

That the integral $I(s)$ of a step function is bounded with respect to the L^1-norm follows because for each

$$s(t) = \sum_{I \in \mathscr{P}} x_I \chi_{R_I}(t) \in S([\mathrm{a}, \mathrm{b}], \mathbb{R})$$

we have

$$|I(s)| = \left| \int_{[\mathrm{a},\mathrm{b}]} s \right| = \left| \sum_{I \in \mathscr{P}} x_I \lambda(R_I) \right| \leq \sum_{I \in \mathscr{P}} |x_I| \lambda(R_I)$$

and

$$\|s\|_1 = \int_{[\mathrm{a},\mathrm{b}]} |s| = \sum_{I \in \mathscr{P}} |x_I| \lambda(R_I)$$

(where the second equality follows from pairwise disjointness of the R_I) so that

$$\|I\| = \sup \left\{ \frac{|I(s)|}{\|s\|_1} : s \in S([\mathrm{a}, \mathrm{b}], \mathbb{R}) \right\} \leq 1.$$

Definition. The Daniell-Lebesgue integral is the unique norm-preserving extension of the bounded linear transformation of the integral on $S([\mathrm{a}, \mathrm{b}], \mathbb{R})$ to a bounded linear transformation on $L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$.

Definition 8.2.6. A function $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ is said to be integrable on $[\mathrm{a}, \mathrm{b}]$ if $f \in L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$.

Nota Bene 8.2.7. This definition of the integrable functions on $[\mathrm{a}, \mathrm{b}]$ is very different from the traditional definition of $L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$, although the two are equivalent. The traditional definition of $L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$ is collection of equivalence classes of equal almost everywhere measurable functions $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ (preimages of open sets are "measurable") that satisfy $\|f\|_1 < \infty$. The definition developed here is that $L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$ is the collection of equivalence classes of equal almost everywhere functions $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ that are the pointwise limit functions of sequences of step functions that are Cauchy in the L^1-norm.

It remains to show that $\mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R})$ is a subspace of $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$.

Proposition 8.2.8. For any $f \in \mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R})$ there holds

$$\|f\|_1 \leq \lambda([\mathrm{a},\mathrm{b}])\|f\|_\infty.$$

The proof of this is HW (Exercise 8.7).

Corollary 8.2.9. If $(f_n)_{n=1}^\infty$ is a sequence in $\mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R})$ converging uniformly to f, then $(f_n)_{n=1}^\infty$ also converges to f in the L^1-norm.

The proof of this is HW (Exercise 8.8).

Corollary 8.2.10. For any normed linear space $(X, \|\cdot\|_X)$, if $T : \mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R}) \to X$ (or $T : S([\mathrm{a},\mathrm{b}],\mathbb{R}) \to X$) is a bounded linear transformation with respect to the L^1-norm, then T is a bounded linear transformation with respect to the L^∞-norm.

Proof. For each nonzero $f \in \mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R})$ we have by Proposition 8.2.8 that $\|f\|_1 \leq \lambda([\mathrm{a},\mathrm{b}])\|f\|_\infty$ which implies

$$\frac{1}{\|f\|_\infty} \leq \frac{\lambda([\mathrm{a},\mathrm{b}])}{\|f\|_1}.$$

We use this to get an upper bound on the L^∞-norm of T:

$$\begin{aligned}
\|T\|_\infty &= \sup\left\{\frac{\|T(f)\|_X}{\|f\|_\infty} : f \in \mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R}), f \neq 0\right\} \\
&\leq \sup\left\{\frac{\lambda([\mathrm{a},\mathrm{b}])\|T(f)\|_X}{\|f\|_1} : f \in \mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R}), f \neq 0\right\} \\
&\leq \lambda([\mathrm{a},\mathrm{b}])\|T\|_1
\end{aligned}$$

where for the last inequality we used $\|T(f)\|_X/\|f\|_1 \leq \|T\|_1$, i.e.,

$$\|T\|_1 = \sup\left\{\frac{\|T(f)\|_X}{\|f\|_1} : f \in \mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R}), f \neq 0\right\}.$$

Since $\|T\|_1 < \infty$ by hypothesis, we have $\|T\|_\infty < \infty$.

The identical argument with $\mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R})$ replaced with $S([\mathrm{a},\mathrm{b}],\mathbb{R})$ holds as well. □

Corollary 8.2.11. Let $\mathscr{J} : \mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R}) \to \mathbb{R}$ be a linear transformation that, when restricted to $S([\mathrm{a},\mathrm{b}],\mathbb{R})$, agrees with the integral $\mathscr{I} : S([\mathrm{a},\mathrm{b}],\mathbb{R}) \to \mathbb{R}$. If $\mathscr{J}$ is bounded with respect to the L^1-norm, then $\mathscr{J}$ is the unique extension of $\mathscr{I}$ to $\mathscr{R}([\mathrm{a},\mathrm{b}],\mathbb{R})$.

The proof of this is HW (Exercise 8.10).