Math 346 Lecture #15
8.4 Monotone Convergence and Integration on Unbounded Domains

The normed linear space $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ is complete in the L^1-norm, which means that any sequence $(f_k)_{k=1}^\infty$ in $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ that is L^1-Cauchy converges in the L^1-norm to a function $f \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$, and by the continuity of the integral as a bounded linear transformation from $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ to $\mathbb{R}$ we have the exchanging of limit and integration,

$$\lim_{k\to\infty} \int_{[\mathrm{a},\mathrm{b}]} f_k = \int_{[\mathrm{a},\mathrm{b}]} \lim_{k\to\infty} f_k = \int_{[\mathrm{a},\mathrm{b}]} f.$$

What can we say about a sequence $(f_k)_{k=1}^\infty$ in $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ that converges only pointwise a.e. on $[\mathrm{a},\mathrm{b}]$ to $f : [\mathrm{a},\mathrm{b}] \to \mathbb{R}$?

1. Does the limit function f belong to $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$?
2. And if the limit function f belongs to $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$, does it hold that

$$\lim_{k\to\infty} \int_{[\mathrm{a},\mathrm{b}]} f_k = \int_{[\mathrm{a},\mathrm{b}]} \lim_{k\to\infty} f_k = \int_{[\mathrm{a},\mathrm{b}]} f?$$

The Monotone Convergence Theorem gives sufficient conditions by which these two questions both have an affirmative answer.

8.4.1 Some Basic Integral Properties

We present without proof (as the proofs are given in Chapter 9) some of the basic properties of the Daniell-Lebesgue integral.

Definition 8.4.1. For any nonempty set A and any function $f : A \to \mathbb{R}$ we define the positive and negative parts of f to be, respectively,

$$f^+(a) = \begin{cases} f(a) & \text{if } f(a) \geq 0, \\ 0 & \text{if } f(a) < 0, \end{cases} \quad \text{and} \quad f^-(a) = \begin{cases} -f(a) & \text{if } f(a) < 0, \\ 0 & \text{if } f(a) \geq 0. \end{cases}$$

Note. The positive and negative parts of f satisfy $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Proposition 8.4.2. For any $f, g \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$, the following hold.

(i) If $f \leq g$ a.e. on $[\mathrm{a},\mathrm{b}]$, then
$$\int_{[\mathrm{a},\mathrm{b}]} f \leq \int_{[\mathrm{a},\mathrm{b}]} g.$$

(ii) $$\left| \int_{[\mathrm{a},\mathrm{b}]} f \right| \leq \int_{[\mathrm{a},\mathrm{b}]} |f| = \|f\|_1.$$

(iii) The functions $\max\{f, g\}$, $\min\{f, g\}$, f^+, f^-, and $|f|$ belong to $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$.

(iv) For a measurable $h : [\mathrm{a},\mathrm{b}] \to \mathbb{R}$, if $|h| \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$, then $h \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$.

(v) For $h : [\mathrm{a},\mathrm{b}] \to \mathbb{R}$ measurable, if $\|h\|_\infty < \infty$, then $fh \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ with
$$\|fh\|_1 \leq \|h\|_\infty \|f\|_1.$$

8.4.2 Monotone Convergence

Unfortunately, not every pointwise a.e. convergent sequence of integrable functions converges in the L^1-norm, as illustrated next.

Unexample. Consider the sequence of functions

$$f_k(t) = \begin{cases} k & \text{if } 0 \le t \le 1/k, \\ 1/t & \text{if } \ 1/k < t \le 1. \end{cases}$$

For each k the function f_k is continuous, so that $f_k \in L^1([0,1],\mathbb{R})$.

The sequence $(f_k(t))_{k=1}^\infty$ converges pointwise to $f(t) = 1/t$ for $t \in (0,1]$, and $(f_k(0))_{k=1}^\infty$ does not converge.

Thus $(f_k)_{k=1}^\infty$ converges to f a.e. on $[0,1]$.

But the sequence $(f_k)_{k=1}^\infty$ is not L^1-Cauchy because for $m > n$ we have

$$\begin{aligned} \int_{[0,1]} |f_m - f_n| &= \int_0^{1/m} (m-n)\ dt + \int_{1/m}^{1/n} \left(\frac{1}{t} - n\right)\ dt \\ &= \frac{m-n}{m} + \log\left(\frac{1}{n}\right) - \log\left(\frac{1}{m}\right) - n\left(\frac{1}{n} - \frac{1}{m}\right) \\ &= 1 - \frac{n}{m} - \log n + \log m - 1 + \frac{n}{m} \\ &= \log m - \log n, \end{aligned}$$

which can be made arbitrary large when m is much bigger than n.

The limit function f does not belong to $L^1([0,1],\mathbb{R})$ because $\|f\|_1 = \infty$.

Even when a sequence of integrable functions converges pointwise a.e. to an integrable function, we can not expect that we can exchange the limit and the integral, as illustrated next.

Unexample (slightly different from 8.4.3). Consider the sequence of functions

$$f_k(t) = \begin{cases} 2^k & \text{if } t \in [0, 2^{-k}), \\ 0 & \text{if } t \in [2^{-k}, 1]. \end{cases}$$

Each f_k is a (right-continuous) step function, and thus belongs to $L^1([0,1],\mathbb{R})$.

The sequence $(f_k(t))_{k=1}^\infty$ converges to $f(t) = 0$ for $t \in (0,1]$, while $(f_k(0))_{k=1}^\infty$ diverges.

Thus $(f_k)_{k=1}^\infty$ converges pointwise a.e. to f on $[0,1]$, and the limit function f belongs to $L^1([0,1],\mathbb{R})$.

But for all k there holds

$$\int_{[0,1]} f_k = 1,$$

so that

$$\lim_{k\to\infty} \int_{[0,1]} f_k = 1,$$

while

$$\int_{[0,1]} \lim_{k\to\infty} f_k = \int_{[0,1]} f = 0.$$

Thus we cannot exchange the limit and the integral for this sequence.

The sequence $(f_k)_{k=1}^\infty$ is not L^1-Cauchy because for $l > k$ we have

$$\begin{aligned}
\int_{[0,1]} |f_l - f_k| &= \int_0^{1/2^l} (2^l - 2^k) + \int_{1/2^l}^{1/2^k} 2^k \\
&= \frac{2^l - 2^k}{2^l} + 2^k\left(\frac{1}{2^k} - \frac{1}{2^l}\right) \\
&= 2\left(1 - \frac{2^k}{2^l}\right),
\end{aligned}$$

which equals 1 when $l = k+1$ no matter how large k is.

The two unexamples illustrate what can go wrong with pointwise a.e. convergence and the Daniell-Lebesgue integral, but also hint at what conditions on the sequence may guarantee the limit function is L^1 and the limit and the integral can be exchanged.

Definition 8.4.4. We say a sequence $(f_k)_{k=1}^\infty$ on the domain $[\mathrm{a},\mathrm{b}]$ and codomain $\mathbb{R}$ is monotone increasing if for all $\mathrm{x} \in [\mathrm{a},\mathrm{b}]$ and all $k \in \mathbb{N}$ there holds

$$f_k(\mathrm{x}) \le f_{k+1}(\mathrm{x}).$$

We say that $(f_k)_{k=1}^\infty$ is almost everywhere monotone increasing, denoted by $f_k \le f_{k+1}$ a.e. on $[\mathrm{a},\mathrm{b}]$, if for every $k \in \mathbb{N}$ the set

$$\{\mathrm{x} \in [\mathrm{a},\mathrm{b}] : f_k(\mathrm{x}) > f_{k+1}(\mathrm{x})\}$$

is a set of measure zero. [Recall that the countable union of sets of measure zero is a set of measure zero.]

Monotone decreasing and almost everywhere monotone decreasing are defined analogously.

Remarks. The sequence $(f_k)_{k=1}^\infty$ in the first unexample in monotone increasing but has the property that

$$\lim_{k\to\infty} \int_{[0,1]} f_k = \infty.$$

The sequence $(f_k)_{k=1}^\infty$ in the second unexample has the property that

$$\int_{[0,1]} f_k \le 1 \text{ for all } k \in \mathbb{N},$$

but is not monotone increasing (as stated in Remark 8.4.6).

Theorem 8.4.5 (Monotone Convergence Theorem). If $(f_k)_{k=1}^\infty \subset L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ is almost everywhere increasing and there exists $M \in \mathbb{R}$ such that for all $k \in \mathbb{N}$ there holds

$$\int_{[\mathrm{a},\mathrm{b}]} f_k \le M,$$

then $(f_k)_{k=1}^{\infty}$ is L^1-Cauchy, and hence there exists $f \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ such that

$$f_k \to f \text{ a.e. on } [\mathrm{a},\mathrm{b}]$$

and

$$\lim_{k\to\infty} \int_{[\mathrm{a},\mathrm{b}]} f_k = \int_{[\mathrm{a},\mathrm{b}]} \lim_{k\to\infty} f_k = \int_{[\mathrm{a},\mathrm{b}]} f.$$

The same conclusions hold when $(f_k)_{k=1}^{\infty} \subset L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ is almost everywhere monotone decreasing and there exists $M \in \mathbb{R}$ such that

$$\int_{[\mathrm{a},\mathrm{b}]} f_k \geq M \text{ for all } k \in \mathbb{N}.$$

Proof. Proposition 8.4.2 part (i) and the assumed almost everywhere monotone increasing of $(f_k)_{k=1}^{\infty} \subset L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ imply that for all $k \in \mathbb{N}$ that

$$\int_{[\mathrm{a},\mathrm{b}]} f_k \leq \int_{[\mathrm{a},\mathrm{b}]} f_{k+1} \leq M.$$

Thus the sequence of real numbers

$$\left(\int_{[\mathrm{a},\mathrm{b}]} f_k\right)_{k=1}^{\infty}$$

is monotone increasing and bounded above, and therefore converges to say $L \in \mathbb{R}$ (by the completeness of $\mathbb{R}$).

For any $\epsilon > 0$ there then exists $N \in \mathbb{N}$ such that for all $k \geq N$ there holds

$$0 \leq L - \int_{[\mathrm{a},\mathrm{b}]} f_k < \epsilon.$$

This implies for $l > m \geq N$ that

$$\begin{aligned}
\|f_l - f_m\|_1 &= \int_{[\mathrm{a},\mathrm{b}]} |f_l - f_m| \\
&= \int_{[\mathrm{a},\mathrm{b}]} (f_l - f_m) \quad \text{[use monotonicity of } (f_k) \text{ here]} \\
&= \left(L - \int_{[\mathrm{a},\mathrm{b}]} f_m\right) - \left(L - \int_{[\mathrm{a},\mathrm{b}]} f_l\right) \\
&\leq L - \int_{[\mathrm{a},\mathrm{b}]} f_m \\
&< \epsilon.
\end{aligned}$$

This shows that $(f_k)_{k=1}^{\infty}$ is L^1-Cauchy.

This implies the existence of $f \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ such that $f_k \to f$ a.e. on $[\mathrm{a},\mathrm{b}]$, and that

$$\lim_{k\to\infty} \int_{[\mathrm{a},\mathrm{b}]} f_k = \int_{[\mathrm{a},\mathrm{b}]} f.$$

The argument for an almost everywhere decreasing sequence of functions in $L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$ whose integrals are bounded below is similar. $\square$

8.4.3 Integration on Unbounded Domains

We use the Monotone Convergence Theorem to justify the extension of integration of measurable functions on bounded measurable sets to measurable functions on unbounded measurable sets.

Definitions. A sequence of sets $(E_k)_{k=1}^{\infty}$ is called increasing if $E_k \subset E_{k+1}$ for all $k \in \mathbb{N}$.

A subset $A \subset \mathbb{R}^n$ (not assumed bounded) is called measurable if there exists an increasing sequence of bounded measurable sets E_k such that

$$A = \bigcup_{k=1}^{\infty} E_k.$$

For a measurable subset $A \subset \mathbb{R}^n$ (not assumed bounded), a function $f : A \to \mathbb{R}$ is measurable if there exists an increasing sequence of bounded measurable sets E_k and step functions s_k defined on compact n-intervals $[\mathrm{a}_k, \mathrm{b}_k] \supset E_k$ such that $A = \cup E_k$ and $s_k \chi_{E_k} \to f$ pointwise a.e. on A.

Definition 8.4.7. For a measurable $A \subset \mathbb{R}^n$ (not assumed bounded), and a nonnegative measurable $f : A \to \mathbb{R}$ we say f is integrable on A if there exists an increasing sequence of bounded measurable sets $(E_k)_{k=1}^{\infty}$ and $M \in \mathbb{R}$ such that $A = \cup E_k$, each $f\chi_{E_k}$ is integrable on some $[\mathrm{a}_k, \mathrm{b}_k] \supset E_k$, and

$$\int_{E_k} f = \int_{[\mathrm{a}_k,\mathrm{b}_k]} f\chi_{E_k} \leq M \text{ for all } k \in \mathbb{N}.$$

Note. The sequence of real numbers

$$\left(\int_{E_k} f \right)_{k=1}^{\infty}$$

is bounded above by M and is monotone increasing because the sequence $(E_k)_{k=1}^{\infty}$ is increasing and because $f \geq 0$.

Definition 8.4.7 (Continued). We define the integral of a nonnegative function f integrable on A to be

$$\int_A f = \lim_{k\to\infty} \int_{E_k} f.$$

We say a measurable function $g : A \to \mathbb{R}$ is integrable on A if g^+ and g^- are both integrable on A and we define

$$\int_A g = \int_A g^+ - \int_A g^-.$$

We denote $L^1(A;\mathbb{R})$ to be the collection of equivalence classes of integrable functions on A (modulo equality almost everywhere).

Nota Bene 8.4.8. For an unbounded measurable $A \subset \mathbb{R}^n$, you will show in the HW (Exercise 8.18) that a measurable real-valued function g is integrable on A if and only if $|g|$ is integrable on A, and that $L^1(A, \mathbb{R})$ is a normed linear space.

Unexample (in lieu of 8.4.10). The function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(t) = \frac{t}{1+t^2}$$

is continuous and hence integrable on any compact interval.

The positive and negative parts of g are

$$g^+(t) = \begin{cases} t/(1+t^2) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \quad \text{and} \quad g^-(t) = \begin{cases} -t/(1+t^2) & \text{if } t < 0 \\ 0 & \text{if } t \geq 0. \end{cases}$$

We show that neither g^+ nor g^- is integrable on $\mathbb{R}$.

Taking $E_k = [-k, k]$ for $k \in \mathbb{N}$, we have $R = \cup E_k$ and

$$\int_{E_k} g^+ = \int_0^k \frac{t}{1+t^2}\, dt = \frac{1}{2} \log(1+t^2)\Big|_0^k = \frac{\log(1+k^2)}{2}.$$

For the increasing sequence E_k there is no $M \in \mathbb{R}$ such that

$$\int_{E_k} g^+ \leq M \text{ for all } k \in \mathbb{N}$$

because $\log(1+k^2) \to \infty$ as $k \to \infty$.

Could there be another increasing E_k' with $\bigcup E_k' = \mathbb{R}$ for which there is an M such that $\int_{E_k} g^+ \leq M$ for all $k \in \mathbb{N}$?

No, because E_k' is increasing with $\bigcup E_k' = \mathbb{R}$, there will be l such that $E_k = [-k, k] \subset E_l'$, which implies that $\int_{E_k'} g^+ \to \infty$.

Thus $g^+ \notin L^1(\mathbb{R}, \mathbb{R})$.

Similarly $g^- \notin L^1(\mathbb{R}, \mathbb{R})$ because

$$\int_{E_k} g^- = \int_{-k}^0 \frac{-t}{1+t^2}\, dt = \int_0^k \frac{t}{1+t^2}\, dt = \frac{1}{2}\log(1+k^2) \to \infty.$$

These mean that it is not possible to avoid the undefined and unreconcilable situation of

$$\int_{\mathbb{R}} g^+ - \int_{\mathbb{R}} g^- = \infty - \infty = \ ???.$$

Moreover, since $|g| = g^+ + g^-$ we also have

$$\int_{\mathbb{R}} |g| = \int_{\mathbb{R}} g^+ + \int_{\mathbb{R}} g^- = \infty + \infty = \infty,$$

which is another way of saying that $|g| \notin L^1(\mathbb{R}, \mathbb{R})$.

In the definition of integrable for a nonnegative real-valued measurable function over a domain A (not assumed bounded) we made use of an increasing sequence of bounded measurable sets E_k whose union is A. As was illustrated in the previous example, we show that integrability does not depend on the choice of the increasing E_k.

Theorem 8.4.11. Let $A \subset \mathbb{R}^n$ be a measurable set (not assumed bounded) and let f be a nonnegative real-valued measurable function on A. Suppose $(E_k)_{k=1}^{\infty}$ and $(E'_k)_{k=1}^{\infty}$ are increasing sequences of bounded measurable sets for which

$$\bigcup_{k=1}^{\infty} E_k = A = \bigcup_{k=1}^{\infty} E'_k,$$

and $f\chi_{E_k}$ and $f\chi_{E'_k}$ are integrable on respective compact n-intervals containing E_k and E'_k for all $k \in \mathbb{N}$. If there exists $M \in \mathbb{R}$ such that

$$\int_{E_k} f \leq M \text{ for all } k \in \mathbb{N},$$

then there holds

$$\int_{E'_k} f \leq M \text{ for all } k \in \mathbb{N},$$

and

$$\lim_{k\to\infty} \int_{E_k} f = \lim_{k\to\infty} \int_{E'_k} f.$$

Proof. Since each E_k is a bounded measurable set, there exists a compact n-interval $[\mathrm{a}_k, \mathrm{b}_k]$ such that $E_k \subset [\mathrm{a}_k, \mathrm{b}_k]$ and $\chi_{E_k} \in L^1([\mathrm{a}_k, \mathrm{b}_k], \mathbb{R})$.

Similarly for each E'_m there is a compact n-interval $[\mathrm{a}'_m, \mathrm{b}'_m]$ such that $E'_m \subset [\mathrm{a}'_m, \mathrm{b}'_m]$ and $\chi_{E'_m} \in L^1([\mathrm{a}'_m, \mathrm{b}'_m], \mathbb{R})$.

For a fixed k and a fixed m there exists a compact n-interval $[\mathrm{c}, \mathrm{d}]$ that contains both $[\mathrm{a}_k, \mathrm{b}_k]$ and $[\mathrm{a}'_m, \mathrm{b}'_m]$.

By Proposition 8.3.11 the functions χ_{E_k} and $\chi_{E'_m}$ both belong to $L^1([\mathrm{c}, \mathrm{d}], \mathbb{R})$.

The functions χ_{E_k} and $\chi_{E'_m}$ both have ∞-norms equal to 1.

By hypothesis, the functions $f\chi_{E_k}$ and $f\chi_{E'_m}$ both belong to $L^1([\mathrm{c}, \mathrm{d}], \mathbb{R})$.

By Proposition 8.4.2 part (v) the functions $\chi_{E_k}\chi_{E'_m} f$ and $\chi_{E_k}\chi_{E'_m} f$ both belong to $L^1([\mathrm{c}, \mathrm{d}], \mathbb{R})$.

Thus the restriction χ_{E_k} and $\chi_{E_k} f$ to E'_m both belong to $L^1(E'_m, \mathbb{R})$.

Replacing E_k and E'_m with E'_k and E_m respectively, implies by the above argument that $\chi_{E'_k}$ and $f\chi_{E'_k}$ both belong to $L^1(E_m, \mathbb{R})$.

By hypothesis, there exists $M \in \mathbb{R}$ such that for all $k \in \mathbb{N}$ there holds

$$\int_{E_k} f \leq M.$$

Then for all $m \in \mathbb{N}$ there holds

$$\int_{E'_m} f\chi_{E_k} = \int_{E_k} f\chi_{E'_m} \leq \int_{E_k} f \leq M.$$

Since $(E_k)_{k=1}^{\infty}$ is increasing and $f \geq 0$, the sequence $(f\chi_{E_k})_{k=1}^{\infty}$ is monotone increasing. By the Monotone Convergence Theorem we have for all $m \in \mathbb{N}$ that

$$\int_{E'_m} f = \int_{E'_m} \lim_{k\to\infty} f\chi_{E_k} = \lim_{k\to\infty} \int_{E'_m} f\chi_{E_k} \leq M.$$

This gives the first conclusion of the theorem.

Now since $(E_k)_{k=1}^{\infty}$ is increasing and $f \geq 0$, the sequence $(\int_{E_k} f)_{k=1}^{\infty}$ of real numbers is monotone increasing and bounded above, so there exists $L \in \mathbb{R}$ such that

$$L = \lim_{k\to\infty} \int_{E_k} f.$$

We may thus take $M = L$ in the first conclusion to get

$$\int_{E'_m} f \leq M = L.$$

Since $(E'_m)_{m=1}^{\infty}$ is increasing and $f \geq 0$, the sequence $(\int_{E'_m} f)_{m=1}^{\infty}$ is monotone increasing and bounded above, so there exists $L' \in \mathbb{R}$ such that

$$L' = \lim_{m\to\infty} \int_{E'_m} f.$$

Thus we obtain

$$L' = \lim_{m\to\infty} \int_{E'_m} f \leq L.$$

Reversing the roles of E_k and E'_m and taking $M = L'$ gives the inequality $L \leq L'$.

Therefore $L = L'$ which is the second conclusion. $\square$