Math 346 Lecture #16
8.5 Fatou's Lemma and the Dominated Convergence Theorem

The monotone convergence theorem for sequences of L^1 functions is the key to proving two other important and powerful convergence theorems for sequences of L^1 functions, namely Fatou's Lemma and the Dominated Convergence Theorem.

Nota Bene 8.5.1. All three of the convergence theorems give conditions under which a sequence of L^1-functions converging pointwise a.e. is L^1-Cauchy. These conditions, when applied to pointwise convergent sequences of regulated-integrable or Riemann-integrable functions, fail to guarantee that the limit function is regulated-integrable or Riemann-integrable, respectively. A counterexample for all three of the convergence theorems for the Riemann integral is the sequence of bounded functions that converge pointwise to the Dirichlet function on $[0,1]$. For an enumeration $(q_k)_{k=1}^\infty$ of the rational numbers in $[0,1]$, the functions f_k, defined by $f_k(t) = 1$ when $t \in \{q_1, \ldots, q_k\}$ and $f(t) = 0$ otherwise, form a monotone increasing sequence converging to the Dirichlet function f, defined by $f(t) = 1$ if $t \in \mathbb{Q} \cap [0,1]$ and $f(t) = 0$ otherwise. But the Dirichlet function is not Riemann-integrable because any upper sum is 1 while any lower sum is 0. This means that to use the three convergence theorems we must work in an L^1-space of functions.

8.5.1 Fatou's Lemma.

We will tacitly make use of the following facts about infimums and supremums: For nonempty subsets A and B of $\mathbb{R}$, if $A \subset B$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$.

Definition. The limit inferior of a sequence of real numbers $(x_k)_{k=1}^\infty$ is defined to be

$$\liminf_{k\to\infty} x_k = \lim_{k\to\infty} \left(\inf_{m\geq k} x_m \right).$$

The limit exists (it might be infinite) because the sequence $y_k = \inf_{m\geq k} x_k$ is monotone increasing (technically monotone nondecreasing).

The limit superior of $(x_k)_{k=1}^\infty$ is defined to be

$$\limsup_{k\to\infty} x_k = \lim_{k\to\infty} \left(\sup_{m\geq k} x_k \right).$$

The limit exists (it might be infinite) because the sequence $z_k = \sup_{m\geq k} x_k$ is monotone decreasing (technically monotone nonincreasing).

The liminf and limsup of a sequence of real numbers aways exist even when the limit of the sequence does not, and there always holds $\limsup_{k\to\infty} a_k \geq \liminf_{k\to\infty} a_k$.

A sequence $(a_k)_{k=1}^\infty$ of real numbers converges to $a \in \mathbb{R}$ if and only if

$$\liminf_{k\to\infty} a_k = a = \limsup_{k\to\infty} a_k.$$

Also

$$\liminf_{k\to\infty}(-a_k) = \limsup_{k\to\infty} a_k \quad \text{and} \quad \limsup_{k\to\infty}(-a_k) = \liminf_{k\to\infty} a_k.$$

Definition. For a sequence of real-valued functions $(f_k)_{k=1}^{\infty}$ with domain E, we define $\liminf_{k\to\infty} f_k$ to be the function on E defined by

$$\left(\liminf_{k\to\infty} f_k\right)(t) = \liminf_{k\to\infty} f_k(t)$$

with the possibility of $\liminf_{k\to\infty} f_k(t)$ being infinite for some (or all) $t \in E$.

The function $\limsup_{k\to\infty} f_k$ of sequence of real-valued functions $(f_k)_{k=1}^{\infty}$ is defined similarly, and can be infinite for some or all $t \in E$.

Definition. A measurable function f with domain $[\mathrm{a}, \mathrm{b}]$ and codomain $\mathbb{R}$ is said to be almost everywhere nonnegative, written $f \geq 0$ a.e. on $[\mathrm{a}, \mathrm{b}]$, if the set

$$\{\mathrm{x} \in [\mathrm{a}, \mathrm{b}] : f(\mathrm{x}) < 0\}$$

have measure zero.

Lemma. For a sequence $(f_k)_{k=1}^{\infty}$ of almost everywhere nonnegative integrable functions with domain $[\mathrm{a}, \mathrm{b}]$, the function $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ defined by

$$f(\mathrm{x}) = \inf\{f_k(\mathrm{x}) : k \in \mathbb{N}\}$$

is almost everywhere nonnegative and integrable.

Proof. For each $k \in \mathbb{N}$ define the function $g_k : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$ by

$$g_k(\mathrm{x}) = \min\{f_1(\mathrm{x}), \ldots, f_k(\mathrm{x})\}.$$

Since each of $f_1, \ldots, f_k$ is almost every nonnegative, the function g_k is also almost everywhere nonnegative.

By Proposition 8.4.2 part (iii) and induction, the function g_k is integrable.

Thus for all $k \in \mathbb{N}$ we have

$$\int_{[\mathrm{a},\mathrm{b}]} g_k \geq 0.$$

Since $\min\{f_1(\mathrm{x}), \ldots, f_k(\mathrm{x})\} \geq \min\{f_1(\mathrm{x}), \ldots, f_k(\mathrm{x}), f_{k+1}(\mathrm{x})\}$, the sequence $(g_k)_{k=1}^{\infty}$ is monotone decreasing (technically monotone nonincreasing).

By the Monotone Convergence Theorem, we have

$$\lim_{k\to\infty} g_k \in L^1([\mathrm{a}, \mathrm{b}], \mathbb{R}).$$

Since for all $\mathrm{x} \in [\mathrm{a}, \mathrm{b}]$ we have

$$f(\mathrm{x}) = \inf_{l\in\mathbb{N}} f_l(\mathrm{x}) = \lim_{k\to\infty} g_k(\mathrm{x})$$

we obtain the integrability of f.

Since each g_k is almost everywhere nonnegative, and the countable union of sets of measure zero is a set of measure zero, we have $f \geq 0$ almost everywhere on $[\mathrm{a}, \mathrm{b}]$. $\square$

Theorem 8.5.2 (Fatou's Lemma). For a sequence $(f_k)_{k=1}^{\infty}$ of integrable functions on $[\mathrm{a},\mathrm{b}]$ that are almost everywhere nonnegative, if

$$\liminf_{k\to\infty}\int_{[\mathrm{a},\mathrm{b}]} f_k < \infty,$$

then

(i) $\liminf\limits_{k\to\infty} f_k \in L^1([\mathrm{a},\mathrm{b}],\mathbb{R})$, and

(ii) $\displaystyle\int_{[\mathrm{a},\mathrm{b}]} \liminf_{k\to\infty} f_k \le \liminf_{k\to\infty}\int_{[\mathrm{a},\mathrm{b}]} f_k$.

Proof. For each $k\in\mathbb{N}$ define the almost every nonnegative function $h_k : [\mathrm{a},\mathrm{b}] \to \mathbb{R}$ by

$$h_k(\mathrm{x}) = \inf_{l\ge k} f_l(\mathrm{x}).$$

Since each f_l is integrable and almost everywhere nonnegative the Lemma implies that every h_k is almost everywhere nonnegative and integrable.

The sequence $(h_k)_{k=1}^{\infty}$ is monotone increasing (technically monotone nondecreasing) with

$$\lim_{k\to\infty} h_k = \liminf_{l\to\infty} f_l.$$

Because $(h_k)_{k=1}^{\infty}$ is monotone increasing we have

$$\lim_{k\to\infty} h_k = \liminf_{k\to\infty} h_k.$$

For each $k\in\mathbb{N}$ there holds $h_k(\mathrm{x}) = \inf\{f_l(\mathrm{x}) : l \ge k\} \le f_k(\mathrm{x})$ for all $\mathrm{x}\in[\mathrm{a},\mathrm{b}]$.

Since h_k and f_k are both integrable we have Proposition 8.4.2 part (i) for all $k\in\mathbb{N}$ that

$$\int_{[\mathrm{a},\mathrm{b}]} h_k \le \int_{[\mathrm{a},\mathrm{b}]} f_k.$$

These imply that

$$\liminf_{k\to\infty}\int_{[\mathrm{a},\mathrm{b}]} h_k \le \liminf_{k\to\infty}\int_{[\mathrm{a},\mathrm{b}]} f_k.$$

[This uses the property that if $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ are two sequences of real numbers such that $a_k \le b_k$ for all $k\in\mathbb{N}$, then $\liminf_{k\to\infty} a_k \le \liminf_{k\to\infty} b_k$ (and $\limsup_{k\to\infty} a_k \le \limsup_{k\to\infty} b_k$).]

Since each h_k is integrable and $(h_k)_{k=1}^{\infty}$ is monotone increasing, we have by Proposition 8.4.2 part (i) that for all $k \ge n$ there holds

$$\int_{[\mathrm{a},\mathrm{b}]} h_n \le \int_{[\mathrm{a},\mathrm{b}]} h_k.$$

This implies for each $n\in\mathbb{N}$ that

$$\int_{[\mathrm{a},\mathrm{b}]} h_n \le \inf_{k\ge n}\int_{[\mathrm{a},\mathrm{b}]} h_k,$$

from whence we get

$$\int_{[\mathrm{a,b}]} h_n \leq \liminf_{k\to\infty} \int_{[\mathrm{a,b}]} h_k.$$

By hypothesis there exists $M \in \mathbb{R}$ such that

$$\liminf_{k\to\infty} \int_{[\mathrm{a,b}]} f_k \leq M.$$

Thus we obtain for all $n \in \mathbb{N}$ that

$$\int_{[\mathrm{a,b}]} h_n \leq \liminf_{k\to\infty} \int_{[\mathrm{a,b}]} h_k \leq \liminf_{k\to\infty} \int_{[\mathrm{a,b}]} f_k \leq M.$$

The sequence $(h_n)_{n=1}^{\infty}$ satisfies the hypotheses of the Monotone Convergence Theorem, and we obtain the integrability of

$$\lim_{n\to\infty} h_n = \liminf_{n\to\infty} h_n = \liminf_{n\to\infty} f_n$$

and

$$\lim_{n\to\infty} \int_{[\mathrm{a,b}]} h_n = \int_{[\mathrm{a,b}]} \lim_{n\to\infty} h_n.$$

Putting all the pieces together we obtain

$$\begin{aligned}
\int_{[\mathrm{a,b}]} \liminf_{n\to\infty} f_n &= \int_{[\mathrm{a,b}]} \lim_{n\to\infty} h_n \\
&= \lim_{n\to\infty} \int_{[\mathrm{a,b}]} h_n \\
&\leq \lim_{n\to\infty} \inf_{k\geq n} \int_{[\mathrm{a,b}]} h_k \\
&\leq \lim_{n\to\infty} \inf_{k\geq n} \int_{[\mathrm{a,b}]} f_k \\
&= \liminf_{n\to\infty} \int_{[\mathrm{a,b}]} f_n.
\end{aligned}$$

This completes the proof. $\square$

Remark 8.5.3. To see why Fatou's Lemma holds, consider two nonnegative integrable functions f_0 and f_1 on a compact interval $[a,b]$ of $\mathbb{R}$.

Because

$$\inf\{f_0, f_1\} = \min\{f_0, f_1\} \leq f_0 \quad \text{and} \quad \inf\{f_0, f_1\} = \min\{f_0, f_1\} \leq f_1,$$

and because $\min\{f_0, f_1\}$ is integrable by Proposition 8.4.2. part (iii), we by part Proposition 8.4.2 part (i) that

$$\int_{[a,b]} \inf\{f_0, f_1\} \leq \int_{[a,b]} f_0 \quad \text{and} \quad \int_{[a,b]} \inf\{f_0, f_1\} \leq \int_{[a,b]} f_1,$$

which implies that

$$\int_{[a,b]} \inf\{f_0, f_1\} \le \inf\left\{\int_{[a,b]} f_0, \int_{[a,b]} f_1\right\}.$$

Fatou's Lemma is the analogous result for sequences of integrable almost everywhere nonnegative functions.

Example (in lieu of 8.5.4). There are sequences of functions for which strict inequality holds in Fatou's Lemma.

The sequence of nonnegative integrable functions $(f_n)_{n=1}^\infty$ on the domain $[0,1]$ defined by $f_n = \chi_{[0,1/2)}$ for n odd and $f_n = \chi_{[1/2,1]}$ for n even, satisfies

$$\liminf_{n\to\infty} \int_{[0,1]} f_n = 1/2$$

because the integral of f_n is $1/2$ for every n.

Since $\liminf f_n(t) = 0$ for every $t \in [0,1]$ we get

$$\int_{[0,1]} \liminf_{n\to\infty} f_n = 0 < \frac{1}{2} = \liminf_{n\to\infty} \int_{[0,1]} f_n.$$

8.5.2 Dominated Convergence

We will use Fatou's Lemma to obtain the dominated convergence theorem of Lebesgue. This convergence theorem does not require monotonicity of the sequence $(f_k)_{k=1}^\infty$ of integrable functions, but only that there is an L^1 function g that dominates the pointwise a.e. convergent sequence $(f_k)_{k=1}^\infty$, i.e., $|f_k| \le g$ for all k.

Theorem 8.5.5 (Dominated Convergence Theorem). Suppose $(f_k)_{k=1}^\infty$ is a sequence of real-valued integrable functions on the domain [a, b] that converges pointwise almost everywhere to a function $f : [\mathrm{a}, \mathrm{b}] \to \mathbb{R}$. If there exists $g \in L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$ such that $|f_k| \le g$ a.e. on [a, b] for all k (this requires that g be nonnegative almost everywhere), then $f \in L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$ and

$$\lim_{k\to\infty} \int_{[\mathrm{a},\mathrm{b}]} f_k = \int_{[\mathrm{a},\mathrm{b}]} \lim_{k\to\infty} f_k = \int_{[\mathrm{a},\mathrm{b}]} f.$$

Proof. The hypothesis $|f_k| \le g$ a.e. on [a, b] for all k implies that the functions $h_k = g - f_k$ are nonnegative almost everywhere.

Also by hypotheses, the functions f_k and g are integrable so the functions h_k are integrable as well.

Since $g \in L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$, i.e., $\|g\|_1 < \infty$, we obtain for all $k \in \mathbb{N}$ that

$$\int_{[\mathrm{a},\mathrm{b}]} h_k = \int_{[\mathrm{a},\mathrm{b}]} (g - f_k) = \int_{[\mathrm{a},\mathrm{b}]} |g - f_k| \le \int_{[\mathrm{a},\mathrm{b}]} \big(|g| + |f_k|\big) \le \int_{[\mathrm{a},\mathrm{b}]} 2g \le 2\|g\|_1 < \infty.$$

Thus the sequence $(h_k)_{k=1}^\infty$ satisfies the hypotheses of Fatou's Lemma, and we get the integrability of

$$g - f = g - \lim_{k\to\infty} f_k = \lim_{k\to\infty} (g - f_k) = \liminf_{k\to\infty} (g - f_k) = \liminf_{k\to\infty} h_k.$$

This implies the integrability of f because

$$f = g - \liminf_{k\to\infty} h_k$$

which is a linear combination of two integrable functions and hence integrable.

By the other conclusion of Fatou's Lemma we have

$$\begin{aligned}
\int_{[a,b]} g - \int_{[a,b]} f &= \int_{[a,b]} (g - f) \\
&= \int_{[a,b]} \left(g - \lim_{k\to\infty} f_k\right) \\
&= \int_{[a,b]} \left(g - \limsup_{k\to\infty} f_k\right) \\
&= \int_{[a,b]} \liminf_{k\to\infty} (g - f_k) \\
&= \int_{[a,b]} \liminf_{k\to\infty} h_k \\
&\leq \liminf_{k\to\infty} \int_{[a,b]} h_k \\
&= \liminf_{k\to\infty} \int_{[a,b]} (g - f_k) \\
&= \int_{[a,b]} g - \limsup_{k\to\infty} \int_{[a,b]} f_k.
\end{aligned}$$

Since the integral of g is finite, we can cancel it from both sides to get

$$-\int_{[a,b]} f \leq -\limsup_{k\to\infty} \int_{[a,b]} f_k.$$

which yields

$$\limsup_{k\to\infty} \int_{[a,b]} f_k \leq \int_{[a,b]} f.$$

Applying the above argument to the sequence of functions $\tilde{h}_k = g + f_k$ results in the inequality

$$\int_{[a,b]} f \leq \liminf_{k\to\infty} \int_{[a,b]} f_k.$$

Since the liminf is always smaller or equal to the limsup of a sequence, we have that

$$\lim_{k\to\infty} \int_{[a,b]} f_k$$

exists and is equal to $\int_{[a,b]} f = \int_{[a,b]} \lim_{k\to\infty} f_k$. $\square$

Example (in lieu of 8.5.7). For any $T > 0$, can you guess the value of

$$\lim_{n\to\infty} \int_0^T \sin(1 - e^{-x/n})e^{-x/2}\ dx?$$

There is little hope of finding an explicit antiderivative of the integrand and applying the Fundamental Theorem of Calculus.

Instead we consider the pointwise limit function of the sequence

$$f_n(x) = \sin(1 - e^{-x/n})e^{-x/2}.$$

By continuity we have for all $x \in [0, T]$ that

$$\lim_{n\to\infty} \sin(1 - e^{-x/n})e^{-x/2} = \sin(1 - e^0)e^{-x/2} = 0.$$

This implies that

$$\int_0^T \lim_{n\to\infty} \sin(1 - e^{-x/n})e^{-x/2}\ dx = 0.$$

If we can find an L^1 function $g : [0, T] \to \mathbb{R}$ that dominates $(f_n)_{n=1}^\infty$, then we have by the Dominated Convergence Theorem that

$$\lim_{n\to\infty} \int_0^T \sin(1 - e^{-x/n})e^{-x/2}\ dx = \int_0^T \lim_{n\to\infty} \sin(1 - e^{-x/n})e^{-x/2}\ dx = 0.$$

Since

$$|f_n(x)| \leq |\sin(1 - e^{-x/n})e^{-x/2}| \leq e^{-x/2}.$$

a candidate for the dominating function is the continuous

$$g(x) = e^{-x/2} \geq 0.$$

This candidate belongs to $L^1([0, T], \mathbb{R})$ for all $T > 0$ because

$$\|g\|_1 = \int_0^T g\ dx = \int_0^T e^{-x/2}\ dx = \left[\frac{e^{-x/2}}{-1/2}\right]_0^T = 2 - 2e^{-T/2} \leq 2.$$

Thus by the Dominated Convergence Theorem we have

$$\lim_{n\to\infty} \int_0^T \sin(1 - e^{-x/n})e^{-x/2}\ dx = 0.$$

The Dominated Convergence Theorem is used to prove the following result about convergence and integration of a series.

Proposition 8.5.8. If $(f_k)_{k=1}^\infty \subset L^1([\mathrm{a}, \mathrm{b}], \mathbb{R})$ satisfies

$$\sum_{k=1}^\infty \int_{[\mathrm{a},\mathrm{b}]} |f_k| < \infty,$$

then the series

$$\sum_{k=1}^{\infty} f_k$$

converges pointwise a.e. on $[\mathrm{a}, \mathrm{b}]$ to an integrable function and

$$\int_{[\mathrm{a},\mathrm{b}]} \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_{[\mathrm{a},\mathrm{b}]} f_k.$$

Proof. Suppose

$$\sum_{k=1}^{\infty} \int_{[\mathrm{a},\mathrm{b}]} |f_k| < \infty.$$

This implies (by HW, Exercise 8.16) that for almost every $\mathrm{x} \in [\mathrm{a}, \mathrm{b}]$ the series

$$\sum_{k=1}^{\infty} |f_k(\mathrm{x})|$$

converges, and the resulting series

$$\sum_{k=1}^{\infty} |f_k| \in L^1([\mathrm{a}, \mathrm{b}], \mathbb{R}).$$

The convergence for almost all $\mathrm{x} \in [\mathrm{a}, \mathrm{b}]$ of $\sum_{k=1}^{\infty} |f_k(\mathrm{x})|$ (this is absolute convergence) implies by Proposition 5.6.13 that for almost all $\mathrm{x} \in [\mathrm{a}, \mathrm{b}]$ we have convergence of

$$\sum_{k=1}^{\infty} f_k(\mathrm{x}).$$

The partial sums

$$h_k(\mathrm{x}) = \sum_{l=1}^{k} f_l(\mathrm{x})$$

are dominated by the L^1 function $\sum_{k=1}^{\infty} |f_k(\mathrm{x})|$ because

$$|h_k(\mathrm{x})| \leq \sum_{l=1}^{k} |f_l(\mathrm{x})| \leq \sum_{l=1}^{\infty} |f_l(\mathrm{x})|.$$

By the Dominated Convergence Theorem we have

$$\begin{aligned}
\lim_{k\to\infty} \int_{[\mathrm{a},\mathrm{b}]} h_k(\mathrm{x}) &= \lim_{k\to\infty} \int_{[\mathrm{a},\mathrm{b}]} \sum_{l=1}^{k} f_l(\mathrm{x}) = \int_{[\mathrm{a},\mathrm{b}]} \lim_{k\to\infty} \sum_{l=1}^{k} f_l(\mathrm{x}) \\
&= \int_{[\mathrm{a},\mathrm{b}]} \lim_{k\to\infty} h_k(\mathrm{x}) = \int_{[\mathrm{a},\mathrm{b}]} \sum_{l=1}^{\infty} f_l(\mathrm{x}).
\end{aligned}$$

Since the partial sum $h_k(\mathrm{x})$ is a finite sum we have

$$\lim_{k\to\infty} \int_{[\mathrm{a},\mathrm{b}]} h_k(\mathrm{x}) = \lim_{k\to\infty} \int_{[\mathrm{a},\mathrm{b}]} \sum_{l=1}^{k} f_l(\mathrm{x}) = \lim_{k\to\infty} \sum_{l=1}^{k} \int_{[\mathrm{a},\mathrm{b}]} f_k(\mathrm{x}) = \sum_{l=1}^{\infty} \int_{[\mathrm{a},\mathrm{b}]} f_k(\mathrm{x}),$$

which gives the result. $\square$