Math 346 Lecture #19
10.1 Curves and Arclength

We review the theory of smooth curves in $\mathbb{R}^n$, and extend this to the general Banach space setting.

Smooth curves are examples of one-dimensional "manifolds." In this chapter we extend Calculus on Banach spaces to manifolds.

Throughout this lecture we let $(X, \|\cdot\|)$ be a Banach space.

Definition 10.1.1. For an interval $I \subset \mathbb{R}$, a smooth parameterized curve in X is an injective C^1 function $\sigma : I \to X$ for which $D\sigma \in C(I, \mathscr{B}(\mathbb{R}, X))$ satisfies $D\sigma(t) \neq 0$ for all $t \in I$.

Note. We often write σ' in place of $D\sigma$.

Note. No condition is placed on the interval I; it could be compact, open, half-closed half-open, unbounded, etc. We often refer to the domain I as "time" and use the variable t to denote elements of I.

Note. When I is not open, continuity and continuous differentiability of σ at an endpoint c of I is in the one-sided sense, i.e., if c is the left endpoint of I, then

$$\lim_{t \to c^+} \sigma(t) = \sigma(c)$$

and there exists $A \in \mathscr{B}(\mathbb{R}, X)$ such that $A = \lim_{t \to c^+} D\sigma(t)$ and

$$\lim_{t \to c^+} \frac{\|\sigma(t) - \sigma(c) - A(t - c)\|}{t - c} = 0,$$

with similar statements holding if c is the right endpoint of I.

Definition. For a compact interval $[a, b]$, a C^1 function $\sigma : [a, b] \to X$ is called a simple closed curve if $\sigma(a) = \sigma(b)$ and σ is injective on $[a, b)$.

Remark 10.1.2. Some of the results of this section holds for curves $\sigma : I \to X$ when I is closed but σ is not differentiable at the endpoints of I but are continuous on I and C^1 on the interior of I.

Definition. The tangent "vector" of a smooth parameterized curve $\sigma : I \to X$ at $t \in I$ is the map $\sigma'(t) \in \mathscr{B}(\mathbb{R}, X)$.

Using the isomorphism $\mathscr{B}(\mathbb{R}, X) \cong X$ given by sending $\nu \in \mathscr{B}(\mathbb{R}, X)$ to $\nu(1) \in X$ (see Theorem 6.5.4), we can think of $\sigma'(t)$ as a vector in X.

The tangent line of a smooth parameterized curve $\sigma : I \to X$ at $t \in I$ is the map $L : \mathbb{R} \to X$ defined by

$$L(\tau) = \sigma'(t)(\tau) + \sigma(t) = \tau\sigma'(t)(1) + \sigma(t)$$

where we have used the linearity of $\sigma'(t)$, i.e., $\sigma'(t)(\tau) = \sigma'(t)(\tau \cdot 1) = \tau\sigma(t)(1)$.

Replacing $\sigma'(t)(1)$ with $\sigma'(t)$ gives the more familiar looking tangent line

$$L(\tau) = \tau\sigma'(t) + \sigma(t).$$

10.1.1 Parameterizations and Equivalent Curves

The image of a smooth parameterized curve $\sigma : I \to X$ is a curve in X and there may be other smooth parameterized curves with the same image. This leads to two equivalence relations on smooth parameterized curves.

Definition 10.1.4. Two smooth parameterized curves $\sigma_1 : I \to X$ and $\sigma_2 : J \to X$ are called equivalent if there exists C^1 bijection $\phi : I \to J$ with $\phi'(t) > 0$ for all $t \in I$ such that

$$\sigma_2 \circ \phi = \sigma_1.$$

This equivalence is an equivalence relation for which the reflexivity and transitivity are readily verified (see note below for verifying the symmetry condition).

When two smooth parameterized curves are equivalent, we say that $\sigma_2 = \sigma_1 \circ \phi$ is a reparameterization of σ_1 and refer to ϕ as a reparameterization.

The condition $\phi' > 0$ on a reparameterization preserves the orientation of the curve.

Each equivalence class of smooth parameterized curves is called a smooth oriented curve.

Note. A C^1 bijection $\phi : I^\circ \to J^\circ$ with $\phi'(t) > 0$ for all $t \in I^\circ$ is a C^1 diffeomorphism from I° to J° by the Inverse Function Theorem. The notion of diffeomorphism on open intervals extends to closures of open intervals through the one-sided limits. Thus we may speak of diffeomorphisms of intervals that are not open. This extension of diffeomorphism is needed when verifying the symmetry condition for the equivalence of smooth parameterized curves to be an equivalence relation.

Definition. Another equivalence relation on smooth parameterized curves is obtained by replacing $\phi' > 0$ with $\phi' \neq 0$ on reparameterizations. Each equivalence class for this equivalence relation is called a smooth unoriented curve, or simply a smooth curve.

A reparameterization ϕ with $\phi' < 0$ reverses the orientation of a smooth curve.

Remark 10.1.5. The tangent vector $\sigma'(t)$ of a smooth curve $\sigma : I \to X$ at the point $\sigma(t)$ for a $t \in I$ depends on the parameterization σ.

A reparameterization $\phi : I \to J$ of σ leads to a possible different tangent vector at the same point $\sigma(t)$ because for the unique $s \in J$ that satisfies $t = \phi(s)$ we have $\sigma \circ \phi(s) = \sigma(t)$ and

$$\frac{d}{ds}\big(\sigma \circ \phi(s)\big) = \sigma'(\phi(s))\phi'(s) = \sigma'(t)\phi'(s),$$

i.e., the tangent vector $\sigma'(t)$ at $\sigma(t)$ is scaled by the derivative $\phi'(s)$ which derivative may not be 1.

However, the unit tangent vector

$$T(t) = \frac{\sigma'(t)}{\|\sigma'(t)\|}$$

is the same for all orientation preserving reparameterizations of σ and is therefore is well-defined for the equivalence class of a smooth oriented curve.

The unit tangent vector will change sign for an orientation-reversing reparameterzation.

Definition 10.1.6. A finite collection of smooth parameterized curves $\sigma_i : [a_i, b_i] \to X$, $i = 1, \dots, k$, is called a piecewise-smooth parameterized curve if there holds

$$\sigma_i(b_i) = \sigma_{i+1}(a_{i+1}) \text{ for all } i = 1, \dots, k-1.$$

A piecewise smooth parameterized curve is denoted by

$$\sigma_1 + \cdots + \sigma_k$$

and is also known as the concatenation of k smooth parameterized curves.

Remark 10.1.7. Although we focus mainly on smooth parameterized curves, most of the results extend piecewise to piecewise smooth parameterized curves.

10.1.2 Arclength

The arclength of a smooth oriented curve is precisely the length of the curve, a quantity that should be independent of the smooth parameterized curve in the equivalence class of the smooth oriented curve.

Definition 10.1.8. The arclength $\text{len}(\sigma)$ of a smooth parameterized curve $\sigma : [a, b] \to X$ is the quantity

$$\text{len}(\sigma) = \int_{[a,b]} \|\sigma'\| = \int_a^b \|\sigma'(t)\| \, dt.$$

Note. The integrand in the definition of $\text{len}(\sigma)$ is integrable because $\|\sigma'\|$ is the composition of two continuous functions where σ' has compact domain.

Note. The arclength depends on the norm. Difference arclength can result from different norms as illustrated next.

Example (in lieu of 10.1.10). For the Banach space $X = M_3(\mathbb{R})$ with the norm $\|\cdot\|_\infty$, find the arclength of the curve $\sigma : [0, 1] \to X$ given by

$$\sigma(t) = \begin{bmatrix} 1 & e^t & t \\ 0 & 1 & e^{-t} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since

$$\sigma'(t) = \begin{bmatrix} 0 & e^t & 1 \\ 0 & 0 & -e^{-t} \\ 0 & 0 & 0 \end{bmatrix}$$

we have

$$\|\sigma'(t)\|_\infty = e^t + 1 \text{ for all } t \in [0, 1].$$

Thus the arclength of σ is

$$\text{len}(\sigma) = \int_0^1 (1 + e^t) \, dt = \left[t + e^t\right]_0^1 = e.$$

If instead we use the Frobenius norm $\|\cdot\|_F$ on $M_3(\mathbb{R})$, we have

$$\begin{aligned}
\|\sigma'(t)\|_F &= \sqrt{\operatorname{tr}(\sigma'(t)^{\mathrm{T}}\sigma'(t))} \\
&= \left\{ \operatorname{tr}\begin{bmatrix} 0 & 0 & 0 \\ e^t & 0 & 0 \\ 1 & -e^{-t} & 0 \end{bmatrix}\begin{bmatrix} 0 & e^t & 1 \\ 0 & 0 & -e^{-t} \\ 0 & 0 & 0 \end{bmatrix}\right\}^{1/2} \\
&= \left\{ \operatorname{tr}\begin{bmatrix} 0 & 0 & 0 \\ 0 & e^{2t} & e^t \\ 0 & e^t & 1+e^{-2t} \end{bmatrix}\right\}^{1/2} \\
&= \sqrt{e^{2t}+1+e^{-2t}}.
\end{aligned}$$

In the Frobenius norm, the arclength of σ is

$$\begin{aligned}
\operatorname{len}(\sigma) &= \int_0^1 \sqrt{e^{2t}+1+e^{-2t}}\, dt \\
&= \int_0^1 \sqrt{e^{2t}+2+e^{-2t}-1}\, dt \\
&= \int_0^1 \sqrt{(e^t+e^{-t})^2-1}\, dt \\
&= \int_0^1 \sqrt{4\cosh^2(t)-1}\, dt \approx 2.122068737,
\end{aligned}$$

the approximation given by Maple.

Proposition 10.1.12. The arclength of a smooth curve is the same for all parameterizations of it, that is, if $\sigma : [a,b] \to X$ and $\gamma = \sigma \circ \phi$ for a C^1-diffeomorphism $\phi : [c,d] \to [a,b]$ with $\phi'(t) \neq 0$ for all $t \in [c,d]$, then $\operatorname{len}(\sigma) = \operatorname{len}(\gamma)$.

Proof. The continuity of ϕ' and the condition $\phi'(t) \neq 0$ for all $t \in [c,d]$ imply by the Intermediate Value Theorem that either $\phi'(t) > 0$ for all $t \in [c,d]$ or $\phi'(t) < 0$ for all $t \in [c,d]$ (because if $\phi'(t_1) < 0 < \phi'(t_2)$ for some $t_1, t_2 \in [c,d]$, continuity of ϕ' implies by the Intermediate Value Theorem that there exists t_3 in between t_1 and t_2 such that $\phi'(t_3) = 0$, a contradiction).

Suppose $\phi'(t) < 0$ for all $t \in [c,d]$. (The other case is similar.)

Since $\phi : [c,d] \to [a,b]$ is bijective, there exists a unique $t_0 \in [c,d]$ such that $\phi(t_0) = b$.

To show that $t_0 = c$ we suppose that $t_0 > c$.

Then by the Mean Value Theorem there exists $\xi \in (c, t_0)$ for which

$$b - \phi(c) = \phi(t_0) - \phi(c) = \phi'(\xi)(t_0 - c) < 0.$$

This means that $\phi(c) > b$ which is impossible because $\phi(c) \in [a,b]$, i.e., $\phi(c) \leq b$.

This contradiction implies that $t_0 = c$, so that $\phi(c) = b$.

Similarly we obtain $\phi(d) = a$.

By the Change of Variables Theorem, the substitution $u = \phi(t)$ gives

$$\begin{aligned}
\operatorname{len}(\gamma) &= \int_c^d \|\gamma'(t)\|\ dt \\
&= \int_c^d \|D\gamma(t)\|\ dt \quad [D\gamma = \gamma'] \\
&= \int_c^d \|D(\sigma \circ \phi)(t)\|\ dt \quad [\gamma = \sigma \circ \phi] \\
&= \int_c^d \|D\sigma(\phi(t))D\phi(t)\|\ dt \quad [\text{chain rule}] \\
&= \int_c^d \|D\sigma(\phi(t))\|\, |D\phi(t)|\ dt \quad [D\phi(t) \text{ is a scalar}] \\
&= \int_c^d \|\sigma'(\phi(t))\|\, |\phi'(t)|\ dt \\
&= -\int_c^d \|\sigma'(\phi(t))\|\, \phi'(t)\ dt \quad [\phi'(t) < 0] \\
&= -\int_b^a \|\sigma'(u)\|\ dt \quad [\text{change of variable}] \\
&= \int_a^b \|\sigma'(u)\|\ dt \\
&= \operatorname{len}(\sigma).
\end{aligned}$$

This shows that a orientation-reversing reparameterization of the smooth curve does not change its arclength. □

Definition 10.1.13. The arclength function of a smooth parameterized curve $\sigma : [a, b] \to X$ is the function $s : [a, b] \to \mathbb{R}$ defined by

$$s(t) = \operatorname{len}(\sigma|_{[a,t]}) = \int_a^t \|\sigma'(\tau)\|\ d\tau;$$

the range of the arclength function is $[0, \operatorname{len}(\sigma)]$.

Note. The arclength function s is differentiable by part (i) of the Fundamental Theorem of Calculus because the integrand $\|\sigma'(\tau)\|$ is the the composition of two continuous functions and hence continuous; the derivative of the arclength function is

$$s'(t) = \|\sigma'(t)\|.$$

Since $t \to \|\sigma'(t)\|$ is continuous on $[a, b]$, the arclength function is C^1.

Definition. We say that a smooth parameterized curve $\sigma : [a, b] \to X$ is parameterized by arclength if $a = 0$ and $\|\sigma'(t)\| = 1$ for all $t \in [0, b]$.

Note. A smooth curve $\sigma : [0, b] \to X$ parameterized by arclength has the property that

$$s(t) = \int_0^t \|\sigma'(\tau)\|\ d\tau = \int_0^t d\tau = t$$

for all $t \in [0, b]$, and the property that

$$\mathrm{len}(\sigma) = \int_0^b d\tau = b.$$

Example (in lieu of 10.1.14). For $X = M_3(\mathbb{R})$ with norm $\|\cdot\|_\infty$, consider the smooth parameterized curve $\sigma : [0, \pi] \to X$ given by

$$\sigma(t) = \begin{bmatrix} 1 & t & 0 \\ \sin t & 2 & 0 \\ -1 & 0 & \cos t \end{bmatrix}.$$

Since

$$\sigma'(t) = \begin{bmatrix} 0 & 1 & 0 \\ \cos t & 0 & 0 \\ 0 & 0 & -\sin t \end{bmatrix}$$

we have

$$\|\sigma'(t)\|_\infty = 1 \text{ for all } t \in [0, \pi].$$

Thus σ is parameterized by arclength.

Proposition 10.1.15. For a smooth parameterized curve $\sigma : [a, b] \to X$, the arclength function $s : [a, b] \to [0, \mathrm{len}(\sigma)]$ is a diffeomorphism.

The proof of this is HW (Exercise 10.2 where you need only show that the C^1 function s is bijective; that s is a diffeomorphism follows from the Inverse Function Theorem and the one-sided limits.)

Definition 10.1.16. For a smooth parameterized curve $\sigma : [a, b] \to X$ and its arclength function $s : [a, b] \to [0, \mathrm{len}(\sigma)]$, the reparameterization of σ by arclength is the smooth parameterized curve $\gamma : [0, \mathrm{len}(\sigma)] \to X$ given by

$$\gamma = \sigma \circ s^{-1}.$$

Proposition 10.1.17. The reparameterization $\gamma : [0, \mathrm{len}(\sigma)] \to X$ by arclength of a smooth parameterized curve $\sigma : [a, b] \to X$ is a smooth curve parameterized by arclength, i.e.,

$$\|\gamma'(\tau)\| = 1 \text{ for all } \tau \in [0, \mathrm{len}(\sigma)].$$

Proof. For $\tau \in [0, \mathrm{len}(\sigma)]$, set $t = s^{-1}(\tau)$, i.e., $\tau = s(t)$.

By the Inverse Function Theorem the derivative of s^{-1} satisfies

$$Ds^{-1}(\tau) = Ds^{-1}(s(t)) = [Ds(t)]^{-1} = \frac{1}{Ds(t)}.$$

Thus we have

$$D\gamma(\tau) = D(\sigma \circ s^{-1})(\tau) = D\sigma(s^{-1}(\tau))Ds^{-1}(\tau) = \frac{D\sigma(t)}{Ds(t)}$$

Since $D\sigma(t) = \sigma'(t)$ and $Ds(t) = \|\sigma'(t)\|$ the norm of $\gamma'(t) = D\gamma(t)$ is 1 for all $t \in [0, \mathrm{len}(\sigma)]$. □