Math 346 Lecture #20
10.2 Line Integrals

We define the integral of a function over a smooth parameterized curve. This has applications for computing mass of a wire by integrating a mass density function over the curve representing the wire, computing work done by a force in moving a particle along a curve. We will extensively use line integrals in Chapter 11.

Throughout this lecture let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces over the same field $\mathbb{F}$.

10.2.1 Line Integrals

Recall from Section 10.1 that any smooth curve C (including simple closed curves) can be parameterized by arclength. This means that, with L being the arclength of C, there is a smooth parameterization $\gamma : [0, L] \to X$ such that $C = \gamma([0, L]) \subset X$ and $\|\gamma'(s)\|_X = 1$ for all $s \in [0, L]$.

Definition 10.2.1. For a smooth curve or simple closed curve C parameterized by arclength $\gamma : [0, L] \to X$, and a function $f : C \to Y$, the line integral of f over C is the quantity

$$\int_C f \; ds = \int_0^L f(\gamma(s)) \; ds \in Y$$

if the integral exists.

Note. The integral is the regulated integral if Y is not a finite dimensional Banach space. Otherwise the integral is the Daniell-Lebesgue integral component wise when Y is a finite dimensional Banach space. Integrability of $f \circ \gamma$ will be in the respective sense. In both situations the integrability of $f \circ \gamma$ will follow when $f \in C(U, Y)$ for an open $U \subset X$ containing C.

Note. The arclength parameterization of a smooth curve or simple closed curve always exists but may not be easily found. Any smooth parameterization of the curve can be used to compute the line integral, as stated next.

Proposition 10.2.2. For a smooth curve or simple closed curve C with smooth parameterization $\sigma : [a, b] \to X$ and a function $f : C \to Y$ there holds

$$\int_C f \; ds = \int_a^b f(\sigma(t)) \|\sigma'(t)\|_X \; dt \in Y,$$

if the integral exists.

The proof of this is HW (Exercise 10.8).

Remark 10.2.3. Each oriented curve has only one parameterization by arclength. Proposition 10.2.2 implies that the line integral only depends on the oriented class of the curve. Switching the orientation of the curve negates the value of the line integral.

Example (in lieu of 10.2.4). For the Banach space $X = M_3(\mathbb{R})$ with the norm $\|\cdot\|_\infty$, consider the smooth curve C with parameterization $\sigma : [0,1] \to X$ given by

$$\sigma(t) = \begin{bmatrix} 1 & e^t & t \\ 0 & 1 & e^{-t} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since

$$\sigma'(t) = \begin{bmatrix} 0 & e^t & 1 \\ 0 & 0 & -e^{-t} \\ 0 & 0 & 0 \end{bmatrix}$$

we have

$$\|\sigma'(t)\|_\infty = e^t + 1 \text{ for all } t \in [0,1].$$

Consider the continuous function $f : M_3(\mathbb{R}) \to \mathbb{R} = Y$ given by

$$f(A) = A_{12}A_{13}A_{23},$$

which is the product of the $(1,2)$, $(1,3)$, and $(2,3)$ entries of a 3×3 matrix A.

The line integral of f over C is

$$\begin{aligned} \int_C f\ ds &= \int_0^1 f(\sigma(t))\|\sigma'(t)\|\ dt \\ &= \int_0^1 (e^t)(t)(e^{-t})(1+e^t)\ dt \\ &= \int_0^1 t(1+e^t)\ dt \\ &= \left[\frac{t^2}{2} + (te^t - e^t)\right]_0^1 \\ &= \frac{3}{2}. \end{aligned}$$

If we use the norm $\|\cdot\|_1$ on $M_3(\mathbb{R})$ instead of the ∞-norm, we get a possibly different value for the line integral of f over C because

$$\|\sigma'(t)\|_1 = \max\{e^t, 1+e^{-t}\} = \begin{cases} 1+e^{-t} & \text{if } 0 \le t \le \sinh^{-1}(1/2), \\ e^t & \text{if } \sinh^{-1}(1/2) \le t \le 1, \end{cases}$$

where $\sinh^{-1}(1/2)$ comes from $e^t = 1 + e^{-t}$, i.e., $\sinh(t) = (e^t - e^{-t})/2 = 1/2$.

Example 10.2.5. (i) For a smooth curve C parameterized by $\sigma : [a,b] \to X$, the line integral of the continuous function $f(\sigma(t)) = \|\sigma'(t)\|_X$ from $[a,b]$ to $Y = \mathbb{R}$ is

$$\int_C f\ ds = \int_a^b \|\sigma'(t)\|_X\ dt,$$

which is the arclength of C.

(ii) If $\rho : C \to \mathbb{R}$ is the mass density of a wire in the shape of a smooth curve $C \subset \mathbb{R}^n$, i.e., the units of ρ are mass per length, then

$$m = \int_C \rho \, ds$$

is the mass of the wire.

(iii) The center of mass $(\overline{x}_1, \dots, \overline{x}_n)$ of a wire in the shape of a smooth curve $C \subset \mathbb{R}^n$ and density $\rho : C \to \mathbb{R}$ is given by

$$\overline{x}_k = \frac{1}{m} \int_C x_k \rho \, ds, \quad k = 1, \dots, n.$$

10.2.2 Line Integrals of Vector Fields

In this section we consider $X = Y = \mathbb{F}^n$ with the usual inner product $\langle \mathrm{x}, \mathrm{y} \rangle = \mathrm{x}^{\mathrm{H}}\mathrm{y}$.

Definition. For a smooth curve or simple closed curve in $\mathbb{F}^n$, a function $F : C \to \mathbb{F}^n$ is called a vector field on C.

Note. Often the vector field F is defined on an open subset U of $\mathbb{F}^n$ that contains C. Then the restriction of F to C gives a vector field F on C.

Definition. For a constant force $F \in \mathbb{R}^n$ the work done by moving a particle along a line segment $\sigma(t) = t\mathrm{v} + \mathrm{c}$, $t \in [a, b]$, for $\mathrm{v}, \mathrm{c} \in \mathbb{R}^n$, is defined to be

$$W = \langle F, (b-a)\mathrm{v} \rangle = (b-a)\langle F, \mathrm{v} \rangle.$$

For a curve C, not necessarily a line segment, smoothly parameterized by $\sigma : [a, b] \to \mathbb{R}^n$, and for a force $F : C \to \mathbb{R}^n$, not necessarily a constant force, the work done in moving a particle along C is approximately the sum of terms

$$\langle F(\sigma(t)), \Delta\sigma \rangle$$

where $\Delta\sigma$ is a small piece of C that contains the point $\sigma(t)$. Taking the limit as $\Delta\sigma \to 0$ gives an exact value for the work.

Definition 10.2.6. Let C be a smooth curve or simple closed curve in $\mathbb{F}^n$ with smooth parameterization $\sigma : [a, b] \to \mathbb{F}^n$. For a vector field $F : C \to \mathbb{F}^n$, if the function $\langle F(\sigma(t)), \sigma'(t) \rangle$ from $[a, b]$ to $\mathbb{F}$ is integrable (in the L^1-sense), then the line integral of the vector field F over C is

$$\int_C \langle F, d\sigma \rangle = \int_C \langle F, \mathbf{T} \rangle \, ds = \int_a^b \langle F(\sigma(t)), \sigma'(t) \rangle \, dt \in \mathbb{F},$$

where $\mathbf{T} : [a, b] \to \mathbb{F}^n$ is the unit tangent vector of C.

Remark. We often use the notation $\mathrm{x} \cdot \mathrm{y}$ in place of $\langle \mathrm{x}, \mathrm{y} \rangle$, in which case we write the line integral of the vector field F over C as

$$\int_C F \cdot d\sigma = \int_C F \cdot \mathbf{T} \, ds = \int_a^b F(\sigma(t)) \cdot \sigma'(t) \, dt.$$

Remark. Often, the integrability of the function $\langle F(\sigma(t)), \sigma'(t)\rangle$ from $[a,b] \to \mathbb{R}$ follows because F is usually defined on an open set U that contains C and $F \in C(U, \mathbb{F}^n)$.

Remark. For standard coordinates on $\mathbb{F}^n$, if we express

$$\sigma(t) = \mathrm{x}(t) = \big(x_1(t), \ldots, x_n(t)\big)$$

and

$$F(\mathrm{x}) = \big(F_1(\mathrm{x}), \ldots, F_n(\mathrm{x})\big),$$

then the line integral of F over C becomes

$$\int_C F \cdot d\sigma = \int_a^b \sum_{i=1}^n \overline{F_i(\mathrm{x}(t))}\, x_i'(t)\ dt = \sum_{i=1}^n \int_a^b \overline{F_i(\mathrm{x}(t))}\, x_i'(t)\ dt = \sum_{i=1}^n \int_C F_i(\mathrm{x}) \cdot dx_i,$$

where the last dot product is the usual inner product on $\mathbb{F}$, i.e., $x \cdot y = \overline{x} y$.

Remark 10.2.7. When $\mathbb{F} = \mathbb{R}$, then we have

$$\int_C F \cdot d\sigma = \sum_{i=1}^n \int_C F_i(\mathrm{x})\, dx_i,$$

because the usual inner product on $\mathbb{R}$ is $x \cdot y = xy$. The line integral of F over C gives the work done by F in moving a particle along the curve C.

Proposition 10.2.8. The line integral of a vector field F over an oriented smooth curve or simple closed curve C is independent of the smooth parameterization of C: if $\sigma_1 : [a,b] \to C$ and $\sigma_2 : [c,d] \to C$ are smooth parameterizations with the same orientation for C, then

$$\int_a^b F(\sigma_1(t)) \cdot \sigma_1'(t)\ dt = \int_{C_1} F \cdot d\sigma_1 = \int_{C_2} F \cdot d\sigma_2 = \int_c^d F(\sigma_2(t)) \cdot \sigma_2'(t)\ dt.$$

If a reparameterization σ_2 reverses the orientation of σ_1 of C, then

$$\int_C F \cdot d\sigma_2 = -\int_C F \cdot d\sigma_1.$$

The proof of this is HW (Exercise 10.9).

Example (in lieu of 10.2.9). Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the continuous vector field defined by

$$F(x,y) = (x^2 - y^2, xy) = (F_1, F_2).$$

Let C be the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ oriented counter-clockwise.

The piecewise-smooth curve C is the concatenation of four smooth curves:

$$\begin{aligned}
&C_1 \text{ given by } \sigma_1 : [0,1] \to \mathbb{R}^2,\ \sigma_1(t) = (x(t), y(t)) = (t, 0),\\
&C_2 \text{ given by } \sigma_2 : [0,1] \to \mathbb{R}^2,\ \sigma_2(t) = (x(t), y(t)) = (1, t),\\
&C_3 \text{ given by } \sigma_3 : [0,1] \to \mathbb{R}^2,\ \sigma_3(t) = (x(t), y(t)) = (1-t, 1),\\
&C_4 \text{ given by } \sigma_4 : [0,1] \to \mathbb{R}^2,\ \sigma_4(t) = (x(t), y(t)) = (0, 1-t).
\end{aligned}$$

We compute the line integral of the vector field F over each of the four curves:

$$\begin{aligned}
\int_{C_1} F \cdot d\sigma_1 &= \int_0^1 F_1(t,0)(1)\ dt = \int_0^1 t^2\ dt = \frac{1}{3}, \\
\int_{C_2} F \cdot d\sigma_2 &= \int_0^1 F_2(1,t)(1)\ dt = \int_0^1 t\ dt = \frac{1}{2} \\
\int_{C_3} F \cdot d\sigma_3 &= \int_0^1 F_1(1-t,1)(-1)\ dt = -\int_0^1 \big((1-t)^2 - 1\big)\ dt = -\frac{2}{3}, \\
\int_{C_4} F \cdot d\sigma_4 &= \int_0^1 F_2(0,1-t)(-1)\ dt = 0.
\end{aligned}$$

The line integral of the vector field F over the counter-clockwise oriented curve C is the sum of the above four line integrals:

$$\begin{aligned}
\int_C F \cdot d\sigma &= \int_{C_1} F \cdot d\sigma_1 + \int_{C_2} F \cdot d\sigma_2 + \int_{C_3} F \cdot d\sigma_3 + \int_{C_4} F \cdot d\sigma_4 \\
&= \frac{1}{3} + \frac{1}{2} - \frac{2}{3} - 0 \\
&= \frac{1}{6}.
\end{aligned}$$

Example 10.2.10 For an open $U \subset \mathbb{R}^n$, a continuous vector field $F : U \to \mathbb{R}^n$ is called conservative if there exists a C^1 function $\phi : U \to \mathbb{R}$ (called a potential) such that $F = D\phi$ on U.

When a vector field $F : U \to \mathbb{R}^n$ is conservative and the smooth curve or simple closed curve $C \subset U$ has parameterization $\sigma : [a,b] \to C$, then

$$\begin{aligned}
\int_C F \cdot \sigma &= \int_a^b F(\sigma(t)) \cdot \sigma'(t)\ dt \\
&= \int_a^b D\phi(\sigma(t)) \cdot \sigma'(t)\ dt \\
&= \int_a^b D(\phi \circ \sigma)(t) \\
&= (\phi \circ \sigma)(b) - (\phi \circ \sigma)(a)
\end{aligned}$$

by the Fundamental Theorem of Calculus.

This says that the line integral of the conservative vector field $F = D\phi$ depends only on the value of the potential function at the endpoints of the curve; the value of the line integral of F is independent of the smooth curve C in U that connects the endpoints.

Another consequence of a conservative vector field is when C is a simple closed curve, then $\sigma(b) = \sigma(a)$ so that

$$\int_C F \cdot d\sigma = 0.$$

In the previous example, the line integral over the simple closed curve $C = C_1 + C_2 + C_3 + C_4$ did not equal zero, so the vector field $F(x, y) = (x^2 - y^2, xy)$ is not conservative.

The vector field $F(x, y) = (3x^2y - y^2, x^3 - 2xy)$ on $\mathbb{R}^2$ is conservative because it is the derivative of the function $\phi(x, y) = x^3y - xy^2$, i.e.,

$$D\phi(x, y) = (3x^2y - y^2, x^3 - 2xy).$$

The integral of this conservative vector field over any simple closed curve C in $\mathbb{R}^2$ will always be 0.

You might recall from Math 314 for a C^1 vector field $F(x, y) = (F_1(x, y), F_2(x, y)$ that if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

on an open ball U in $\mathbb{R}^2$, then there exists a C^2 function $\phi : U \to \mathbb{R}$ such that $F = D\phi$ on U.