Math 346 Lecture #23
10.5 Green's Theorem

Green's Theorem is an analogue of the Fundamental Theorem of Calculus and provides an important tool not only for theoretic results but also for computations.

Green's Theorem requires a topological notion, called simply connected, which we define by way of an important topological theorem known as the Jordan Curve Theorem.

10.5.1 Jordan Curve Theorem

Recall from 5.9.3 the notion of connectedness for a metric space X: if there are disjoint open subsets U and V such that $X = U \cup V$, then X is called disconnected; if X is not disconnected, then X is connected.

Definition 10.5.1. A connected component of a subset $S \subset \mathbb{F}^n$ is a subset $T \subset S$ such that T is connected and T is not contained in any other connected subset of S.

Example (in lieu of 10.5.2). The subset $\mathbb{R}^2 \setminus \{(x, y) : x^2 + y^2 = 1\}$ consists of two connected components, namely $\{(x, y) : x^2 + y^2 < 1\}$ and $\{(x, y) : x^2 + y^2 > 1\}$. The subset $\{(x, y) : x^2 + y^2 = 1\}$ removed from $\mathbb{R}^2$ is a simple closed curve with smooth parameterization $\alpha : [0, 2\pi] \to \mathbb{R}^2$ given by $\alpha(t) = (\cos t, \sin t)$.

Theorem 10.5.3 (Jordan Curve Theorem). Let γ be a simple closed curve in $\mathbb{R}^2$. The complement $\mathbb{R}^2 \setminus \gamma$ consists of two connected components, one of which is bounded, and the other unbounded, where γ is the topological boundary of each component.

Remark 10.5.4. This theorem and its proof are part of the branch of mathematics known as Algebraic Topology. The proof of the Jordan Curve Theorem is quite difficult.

Definition 10.5.5. For a simple closed curve γ in $\mathbb{R}^2$, we call the bounded component of $\mathbb{R}^2 \setminus \gamma$ the interior of γ, and call the unbounded connected component of $\mathbb{R}^2 \setminus \gamma$ the exterior of γ.

If a point $\mathrm{x} \in \mathbb{R}^2$ lies in the interior of γ we say that x is enclosed by γ, or that it lies within γ.

Remark 10.5.6. The complex plane $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$, a homeomorphism from $\mathbb{R}^2$ to $\mathbb{C}$ being given by

$$(x, y) \to x + iy.$$

This means, because the Jordan Curve Theorem is a topological property, that we can apply the Jordan Curve Theorem to simple closed curves in $\mathbb{C}$, and speak of their interiors and exteriors.

Definition 10.5.7. A subset $U \subset \mathbb{R}^2$ or $U \subset \mathbb{C}$ is said to be simply connected if for any simple closed curve γ that lies in U, every point in the interior of γ belongs to U.

Nota Bene 10.5.8. Practically, a subset U of either $\mathbb{R}^2$ or $\mathbb{C}$ is simply connected if it contains no holes. For example, the open unit disk $\{(x, y) : x^2 + y^2 < 1\}$ is simply connected but the punctured open disk $\{(x, y) : 0 < x^2 + y^2 < 1\}$ is not simply connected (see Unexample 10.5.10).

Example 10.5.9. (i) The whole plane $\mathbb{R}^2$ or $\mathbb{C}$ is simply connected.

(ii) For any $\mathrm{x} \in \mathbb{R}^2$ and any $r > 0$, the open ball $B(\mathrm{x}, r)$ in $\mathbb{R}^2$ is simply connected.

Proposition 10.5.11. The interior of any simple closed curve is simply connected.

Proof. For a simple closed curve $\gamma \subset \mathbb{R}^2$, let U and B be the unbounded and bounded connected components of $\mathbb{R}^2 \setminus \gamma$.

To show that the interior of γ is simply connected, we take a simple closed curve $\sigma \subset B$.

Then σ splits $\mathbb{R}^2 \setminus \gamma$ into an unbounded connected component v and a bounded connected component β.

Since U is connected with $U \cap \sigma = \emptyset$, either $U \subset v$ or $U \subset \beta$.

Since β is bounded and U is unbounded, we cannot have $U \subset \beta$.

This implies that $U \subset v$, whence $\beta \subset v^c \subset U^c = B \cup \gamma$.

Since $\gamma \cap B = \emptyset$, we obtain $\beta \subset B$.

This shows that the interior of γ is simply connected. □

We now give names to the two orientations that a simple closed curve in $\mathbb{R}^2$ can have.

Definition 10.5.12. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a simple closed curve with interior $\Theta \subset \mathbb{R}^2$. For $\gamma(t) = (x(t), y(t))$, we define the left-pointing normal vector $\mathrm{n}(t)$ at $t \in [a, b]$ to be

$$\mathrm{n}(t) = (-y'(t), x'(t)).$$

We say that γ is positively oriented if for all $t \in [a, b]$ there is $\delta > 0$ such that for all $0 < h < \delta$ there holds

$$\gamma(t) + h\mathrm{n}(t) \in \Theta.$$

Remark 10.5.13. In other words, a simple closed curve is positively oriented when Θ always lies to the left of the tangent vector $\gamma'(t)$. This is the same as saying the simple closed curve is traversed in the counterclockwise direction. This notion of positive orientation extends to piecewise smooth simple closed curves. If instead Θ always lies to the right of the tangent vector, then γ has negative orientation, which is to say that traversing the simple closed curve is done in clockwise direction.

10.5.2 Green's Theorem

Green's Theorem holds for bounded simply connected subsets of $\mathbb{R}^2$ whose boundaries are simple closed curves or piecewise simple closed curves. To prove Green's Theorem in this general setting is quite difficult. Instead we restrict attention to "nicer" bounded simply connected subsets of $\mathbb{R}^2$.

Definition 10.5.14. A closed subset $\Delta \subset \mathbb{R}^2$ is called an x-simple region if there is a compact interval $[a, b] \subset \mathbb{R}$ and continuous functions $f, g : [a, b] \to \mathbb{R}$ such that f and g are C^1 on (a, b), and

$$\Delta = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\}.$$

A closed subset $\Delta \subset \mathbb{R}^2$ is called a y-simple region if there is a compact interval $[a, b] \subset \mathbb{R}$ and continuous function $f, g : [c, d] \to \mathbb{R}$ such that f and g are C^1 on (a, b), and

$$\Delta = \{(x, y) : c \leq y \leq d, f(y) \leq x \leq g(y)\}.$$

We say a closed subset Δ of $\mathbb{R}^2$ is a simple region if Δ is both an x-simple region and a y-simple region.

Example. (i) Each compact rectangle $[a,b]\times[c,d]$ in $\mathbb{R}^2$ is a simple region.

(ii) Any closed disk $\overline{B(\mathrm{x},r)}$ is a simple region.

(iii) The intersection of $\overline{B(0,r)}$ with the first quadrant $\{(x,y) : x \geq 0, y \geq 0\}$ is a simple region.

Theorem 10.5.15 (Green's Theorem). Let $\gamma : [a,b] \to \mathbb{R}^2$ be a piecewise-smooth, positively oriented, simple closed curve with interior $\Omega \subset \mathbb{R}^2$ such that $\overline{\Omega} = \Omega \cup \gamma$ is the union of a finite number of simple regions $\Delta_1, \ldots, \Delta_m$ for which the pairwise intersections $\Delta_i \cap \Delta_j$, $i \neq j$, all have measure zero. For an open U containing $\overline{\Omega}$, if $P, Q : U \to \mathbb{R}$ are C^1, then

$$\int_{\overline{\Omega}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_\gamma (P,Q)\cdot d\gamma = \int_\gamma P dx + Q dy.$$

Proof. It suffices to prove the theorem when $\overline{\Omega}$ is a simple region.

Then $\overline{\Omega}$ is an x-simple region and we show that

$$\int_{\overline{\Omega}} \frac{\partial Q}{\partial x} = \int_\gamma Q\ dy.$$

With $\overline{\Omega}$ also being a y-simple region, the proof of

$$-\int_{\overline{\Omega}} \frac{\partial P}{\partial y} = \int_\gamma P\ dx$$

is similar.

With $\overline{\Omega}$ being x-simple, there is a compact $[a,b] \subset \mathbb{R}$ and continuous functions $f, g : [a,b] \to \mathbb{R}$ such that f and g are C^1 on (a,b), and

$$\overline{\Omega} = \{(x,y) : a \leq x \leq b, f(x) \leq y \leq g(x)\}.$$

The boundary γ of $\overline{\Omega}$ is the concatenation of four smooth curves γ_i, $i = 1,2,3,4$, oriented in the positive orientation.

Here $\gamma_1 : [a,b] \to \mathbb{R}^2$ is the bottom of the region and is given by

$$\gamma_1(t) = (t, f(t));$$

here we have

$$\int_{\gamma_1} Q\ dy = \int_a^b Q(t, f(t)) f'(t)\ dt.$$

The curve $\gamma_2 : [0,1] \to \mathbb{R}^2$ is the right side of $\overline{\Omega}$ and is given by

$$\gamma_2(t) = (b, (1-t)f(b) + tg(b));$$

here we have

$$\begin{aligned}\int_{\gamma_2} Q\ dy &= \int_0^1 Q\big(b,(1-t)f(b)+tg(b)\big)(g(b)-f(b))\ dt\\ &= \int_{f(b)}^{g(b)} Q(b,z)\ dz,\end{aligned}$$

where we use the change of variable $z=(1-t)f(b)+tg(b)$.

The curve $\gamma_3:[a,b]\to\mathbb{R}^2$ is the top of $\overline{\Omega}$ and is given by

$$\gamma_3(t)=(b+a-t,g(b+a-t));$$

here we have

$$\begin{aligned}\int_{\gamma_3} Q\ dy &= -\int_a^b Q\big(b+a-t,g(b+a-t)\big)g'(b+a-t)\ dt\\ &= (-1)^2\int_b^a Q(z,g(z))g'(z)\ dz = -\int_a^b Q(z,g(z))g'(z)\ dz,\end{aligned}$$

where we use the change of variable $z=b+a-t$.

The curve $\gamma_4:[0,1]\to\mathbb{R}^2$ is the left of $\overline{\Omega}$ and is given by

$$\gamma_4(t)=(a,(1-t)g(a)+tf(a));$$

here we have

$$\begin{aligned}\int_{\gamma_4} Q\ dy &= \int_0^1 Q\big(a,(1-t)g(a)+tf(a)\big)(f(a)-g(a))\ dt\\ &= \int_{g(a)}^{f(a)} Q(a,z)\ dz = -\int_{f(a)}^{g(a)} Q(a,z)\ dz,\end{aligned}$$

where we use the change of variable $z=(1-t)g(a)+tf(a)$.

The four integral we have computed combine to give

$$\int_\gamma Q\ dy = \sum_{i=1}^4 \int_{\gamma_i} Q\ dy.$$

On the other hand, by Fubini's Theorem, the corollary of Leibniz's Rule, and the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\int_{\overline{\Omega}} \frac{\partial Q}{\partial x} &= \int_a^b \left(\int_{f(x)}^{g(x)} \frac{\partial Q}{\partial x}\ dy\right)\ dx\\ &= \int_a^b \left[\frac{d}{dx}\bigg(\int_{f(x)}^{g(x)} Q(x,y)\ dy\bigg) + f'(x)Q(x,f(x)) - g'(x)Q(x,g(x))\right]\ dx\\ &= \int_{f(b)}^{g(b)} Q(b,y)\ dy - \int_{f(a)}^{g(a)} Q(a,y)\ dy + \int_a^b f'(x)Q(x,f(x))\ dx\\ &\quad - \int_a^b g'(x)Q(x,g(x))\ dx.\end{aligned}$$

Comparing this with the direct computation of the line integral of Q on γ we have agreement. $\square$

Example (in lieu of 10.5.16). Can you evaluate the line integral

$$\int_\gamma (y + xe^x \cos x)dx + (x + \ln(1 + y^2 + e^y) \sin y)dy$$

where γ is unit circle?

The functions

$$P(x, y) = y + xe^x \cos x \text{ and } Q(x, y) = x + \ln(1 + y^2 + e^y) \sin y$$

both belong to $C^\infty(\mathbb{R}^2, \mathbb{R})$.

Thus P and Q are C^1 on an open set U containing γ and its interior Ω.

By Green's Theorem we have

$$\int_\gamma P dx + Q dy = \iint_{\overline{\Omega}} (Q_x - P_y) = \iint_{\overline{\Omega}} (1 - 1) = 0.$$

Remark 10.5.17. The conclusion of Green's Theorem extends to general regions of the plane such as the not simply connected annulus

$$\Omega = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$$

with the positive orientation, namely, counterclockwise on the outer boundary while clockwise on the inner boundary.

The extension of Green's Theorem to Ω is achieved by cutting the annulus with a line passing through the origin, say the x-axis, into two simply connected pieces

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$$

and

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4, y \leq 0\}.$$

In keeping the positive orientation, the straight-line boundary components of Ω_1 are traversed from left to right, while the straight-line boundary components of Ω_2 are traversed from right to left.

This means that the contribution to a line integral over the straight-line boundary components of Ω_1 and Ω_2 cancel, and never need to be computed.

On the other hand, we can apply Green's Theorem to Ω_1 and Ω_2 because the intersection $\overline{\Omega_1} \cap \overline{\Omega_2}$ has measure zero; if γ_i is the piecewise smooth boundary curve of Ω_i, $i = 1, 2$, then

$$\iint_{\overline{\Omega}} Q_x - P_y = \sum_{i=1}^{2} \iint_{\overline{\Omega}_i} Q_x - P_y = \sum_{i=1}^{2} \int_{\gamma_i} P dx + Q dy.$$