Math 346 Lecture #24
11.1 Holomorphic Functions

11.1.1 Differentiation on $\mathbb{C}$

The complex plane $\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$, as a normed vector space over $\mathbb{R}$, is isomorphic to $\mathbb{R}^2$ through the map

$$(x, y) \to x + iy,$$

where the norm on $\mathbb{C}$ is $|x + iy| = \sqrt{x^2 + y^2}$, the 2-norm on $\mathbb{R}^2$.

This means that $\mathbb{C}$ and $\mathbb{R}^2$ have the same topology, so an open set in $\mathbb{C}$ will be the same as an open set in $\mathbb{R}^2$.

The complex plane $\mathbb{C}$ is two-dimensional vector space over $\mathbb{R}$, or a real two-dimensional vector space, because $\{1, i\}$ is a basis for $\mathbb{C}$ over $\mathbb{R}$.

The complex plane is also a one-dimensional normed vector space over $\mathbb{C}$, i.e., the complex number 1 is a basis for $\mathbb{C}$ over $\mathbb{C}$.

Because we can think of $\mathbb{C}$ in two different ways as a normed vector space there are two different notions of derivative.

Let U be an open subset of $\mathbb{C}$, and consider a function $f : U \to X$ where X will be either the real vector space $\mathbb{R}^2$ or the complex vector space $\mathbb{C}$, and let $\mathrm{x}_0 = (x_0, y_0) \in U$ or $z_0 = x_0 + iy_0 \in U$.

When we think of $\mathbb{C}$ as a two-dimensional vector space $\mathbb{R}^2$ over $\mathbb{R}$, the real derivative of $f : U \to \mathbb{R}^2$ at a point $\mathrm{x}_0 = (x_0, y_0) \in U$, if it exists, is the unique $L \in \mathscr{B}(\mathbb{R}^2)$ such that

$$\lim_{\mathrm{h}\to 0} \frac{\|f(\mathrm{x} + \mathrm{h}) - f(\mathrm{x}) - L\mathrm{h}\|_2}{\|h\|_2} = 0,$$

i.e., a bounded linear real operator $L = Df(\mathrm{x})$ on $\mathbb{R}^2$ represented in the standard basis by a 2×2 real matrix $\begin{bmatrix}\partial f/\partial x & \partial f/\partial y\end{bmatrix}$.

The real derivative of f at x_0 requires four real numbers in the 2×2 matrix to describe it.

When we think of $\mathbb{C}$ as a complex vector, the complex derivative of $f : U \to \mathbb{C}$ at a point $z_0 \in U$, if it exists, is the unique $A \in \mathscr{B}(\mathbb{C})$ such that

$$\lim_{\xi\to 0} \frac{\|f(z_0 + \xi) - f(z_0) - A\xi\|_2}{\|\xi\|_2} = 0,$$

i.e., a bounded linear complex operator $A = Df(z_0)$ on $\mathbb{C}$ represented in the standard basis by a 1×1 complex matrix whose entry is

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}.$$

The complex derivative of f at z_0 requires one complex number or equivalently two real numbers to describe it.

The real and complex derivatives of f are different because the real derivatives requires four real numbers while the complex derivative requires two real numbers.

We can extend the notions of real derivative and complex derivative to functions $f : U \to X$ for a complex Banach space X with norm $\|\cdot\|_X$ because a complex Banach space is a real Banach space.

We say that $f : U \to X$ is real differentiable at $\mathrm{x}_0 \in U$ if there exists $L \in \mathscr{B}(\mathbb{R}^2, X)$ such that

$$\lim_{\mathrm{h}\to 0} \frac{\|f(\mathrm{x}+\mathrm{h}) - f(\mathrm{x}) - L\mathrm{h}\|_X}{\|h\|_2} = 0.$$

Definition. For U open in $\mathbb{C}$ and $(X, \|\cdot\|_X)$ a complex Banach space, a function $f : U \to X$ is the complex differentiable at $z_0 \in U$ if there is $A \in \mathscr{B}(\mathbb{C}, X)$ such that

$$\lim_{\xi\to 0} \frac{\|f(z_0+\xi) - f(z_0) - A\xi\|_X}{\|\xi\|_2} = 0.$$

Since $\mathscr{B}(\mathbb{C}, X) \cong X$ via the complex linear map $\phi \to \phi(1)$, we may identify the complex derivative $A = Df(z_0)$ of f at z_0 by

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in X,$$

which is to say that $Df(z_0)$ as complex linear transformation acts as

$$Df(z_0)w = wf'(z_0),$$

the complex scalar multiple of $f'(z_0) \in X$ by $w \in \mathbb{C}$.

Definition 11.1.1. For U open in $\mathbb{C}$ and X a complex Banach space, a function $f : U \to X$ is holomorphic on U if f is complex differentiable at each $z_0 \in U$.

By saying f is holomorphic without specifying the domain U, the standing assumption is that f is holomorphic on $\mathbb{C}$, i.e., $U = \mathbb{C}$.

A function holomorphic on $\mathbb{C}$ is called entire.

11.1.2 The Cauchy-Riemann Equations

As we will see, complex differentiable implies real differentiable, but the converse is false. Part of the reason for this is that the complex linear transformation $f'(z_0)$ is given by complex scalar multiplication $w \to wf'(z_0)$ whereas most real linear transformations from $\mathbb{R}^2$ to X are not given by complex scalar multiplication.

Specifically, a complex linear transformation from $\mathbb{C}$ to X is determined by a choice of $\mathrm{b} \in X$ for $f'(z_0)$, i.e., for $w = x + iy$ we have

$$w \to w\mathrm{b} = wf'(z_0) = (x+iy)\mathrm{b} = x\mathrm{b} + iy\mathrm{b} \in X,$$

whereas for a real linear transformation from $\mathbb{R}^2$ to X is determined by a choice of two elements $\mathrm{b}_1, \mathrm{b}_2 \in X$ for $\begin{bmatrix}\partial f/\partial x & \partial f/\partial y\end{bmatrix}$, i.e., for $(x, y) \in \mathbb{R}^2$ and $\begin{bmatrix}\partial f/\partial x & \partial f/\partial y\end{bmatrix} = \begin{bmatrix}\mathrm{b}_1 & \mathrm{b}_2\end{bmatrix}$ we have

$$(x, y) \to \begin{bmatrix}\partial f/\partial x & \partial f/\partial y\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = x\mathrm{b}_1 + y\mathrm{b}_2 \in X.$$

This real linear transformation defines a complex linear transformation if and only if

$$\mathrm{b}_2 = i\mathrm{b}_1,$$

which in terms of the partial real derivatives of f, is that

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}.$$

This equation is known as the Cauchy-Riemann equation.

In the special case of $X = \mathbb{C}$, we can write

$$f(x, y) = u(x, y) + iv(x, y),$$

so that

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{ and } \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y},$$

whence the Cauchy-Riemann equation becomes the pair of equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Theorem 11.1.4 (Cauchy-Riemann). For open $U \subset \mathbb{C}$ and a complex Banach space X, let $f : U \to X$.

(i) If f is holomorphic on U, then it is real differentiable on U, the partials $\partial f/\partial x$ and $\partial f/\partial y$ exist on U, and the Cauchy-Riemann equation $\partial f/\partial y = i\partial f/\partial x$ holds on U.

(ii) If the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist on U, are continuous on U, and satisfy the Cauchy-Riemann equation $\partial f/\partial y = i\partial f/\partial x$ on U, then f is holomorphic on U. [The book has an extra hypothesis, f is real differentiable on U, but this is implied by the continuous existence of the partial derivatives.]

See the book for the proof.

Note. Corollary 11.1.5 is a bit redundant – it is part (ii) of Theorem 11.1.4 with $X = \mathbb{C}$. A better corollary of 11.1.4 is: if $f : U \to \mathbb{C}$ given by $f(z) = u(x, y) + iv(x, y)$ has $u, v : U \to \mathbb{R}$ being C^1 and satisfying the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then f is holomorphic on U.

Example. Prove that $f(z) = z^2$ is entire.

The book showed this in Example 11.1.2 by direct computation of the complex derivative.

We will use Theorem 11.1.4 part (ii). To this end we have

$$z^2 = (x + iy)^2 = x^2 - y^2 + i(2xy)$$

so that

$$u(x,y) = x^2 - y^2 \text{ and } v(x,y) = 2xy.$$

The functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ are C^1 functions for which

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

By Theorem 11.1.4 part (ii), the function $f(z) = z^2$ is holomorphic on $\mathbb{C}$, i.e., it is entire.

Example 11.1.6. Not every function formed in complex variable notation, such as

$$f(z) = \frac{z}{|z|^2}$$

is holomorphic on some open subset U of $\mathbb{C} \setminus \{0\}$.

Here we have

$$f(z) = \frac{x+iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

so that

$$u(x,y) = \frac{x}{x^2+y^2} \text{ and } v(x,y) = \frac{y}{x^2+y^2}.$$

Then

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} \text{ and } \frac{\partial v}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

do not agree on an open set U in $\mathbb{C} \setminus \{0\}$.

Nor do

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} \text{ and } -\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

agree on an open set U in $\mathbb{C} \setminus \{0\}$.

By the contrapositive of Theorem 11.1.4 part (i), the function f is not holomorphic on any open subset U of $\mathbb{C} \setminus \{0\}$.

Note. The complex plane has the conjugate operation, namely $\overline{x+iy} = x - iy$. For a function $f : U \to \mathbb{C}$ with U open in $\mathbb{C}$, there is the function $\bar{f} : U \to \mathbb{C}$ given by conjugating the output of f, i.e., if $f(x+iy) = u(x,y) + iv(x,y)$, then

$$\bar{f}(x+iy) = u(x,y) - iv(x,y).$$

With this we can form the function

$$|f|^2 = f\bar{f} = (u+iv)(u-iv) = u^2 + v^2 \geq 0,$$

and hence the real-valued function

$$|f| = \sqrt{u^2+v^2}.$$

Proposition 11.1.7. Suppose for an open, path-connected $U \subset \mathbb{C}$, that $f : U \to \mathbb{C}$ is holomorphic on U. If $\bar{f}$ is holomorphic on U or $|f|$ is constant on U, then f is constant on U.

The proof of this HW (Exercise 11.5).