Math 346 Lecture #27
11.4 Cauchy's Integral Formula

The Cauchy Integral Formula is probably the most important result in complex analysis. As a consequence of the Cauchy-Goursat Theorem, the Cauchy Integral Formula has some amazing consequences for holomorphic functions, some of which we will see in this lecture, and some in the next.

Throughout we let $(X, \|\cdot\|_X)$ be a Banach space.

11.4.1 Cauchy's Integral Formula

Theorem 11.4.1 (Cauchy's Integral Formula). For a simply connected open set $U \subset \mathbb{C}$, a holomorphic $f : U \to X$, and a simple closed contour γ lying entirely in U and traversed once in the counterclockwise direction, if z_0 is in the interior of γ, then

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,dz.$$

Proof. By the Cauchy-Goursat Theorem and Remark 11.3.15, we can replace γ with a circle σ centered at z_0 with small enough radius $r > 0$ such that γ lies completely within U:

$$\oint_\sigma \frac{f(z)}{z - z_0}\,dz = \oint_\gamma \frac{f(z)}{z - z_0}\,dz.$$

By Lemma 11.3.5, i.e., $\oint_\sigma 1/(z - z_0)\,dz = 2\pi i$, we have

$$f(z_0) = \frac{f(z_0)}{2\pi i}\oint_\sigma \frac{1}{z - z_0} = \frac{1}{2\pi i}\oint_\sigma \frac{f(z_0)}{z - z_0}\,dz.$$

Using the standard parameterization $\gamma(\theta) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$, we have

$$\begin{aligned}
\left\|\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,dz - f(z_0)\right\|_X &= \left\|\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,dz - \frac{1}{2\pi i}\oint_\gamma \frac{f(z_0)}{z - z_0}\,dz\right\|_X \\
&= \left\|\frac{1}{2\pi i}\oint_\gamma \frac{f(z) - f(z_0)}{z - z_0}\,dz\right\|_X \\
&= \left\|\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{re^{i\theta}}\,ire^{i\theta}\,d\theta\right\|_X \\
&= \left\|\frac{1}{2\pi}\int_0^{2\pi} \left[f(z_0 + re^{i\theta}) - f(z_0)\right]\,d\theta\right\|_X \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \|f(z_0 + re^{i\theta}) - f(z_0)\|_X\,d\theta \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \sup\left\{\|f(z) - f(z_0)\|_X : |z - z_0| = r\right\}\,d\theta \\
&\leq \sup\left\{\|f(z) - f(z_0)\|_X : |z - z_0| = r\right\}.
\end{aligned}$$

Since f is holomorphic on U, it is continuous at z_0; thus for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $|z - z_0| < \delta$, there holds $\|f(z) - f(z_0)\|_X < \epsilon$.

This implies for $0 < r < \delta$ that

$$\sup\{\|f(z) - f(z_0)\|_X : |z - z_0| = r\} < \epsilon.$$

It now follows that

$$\left\| \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz - f(z_0) \right\|_X \leq \epsilon.$$

Since this holds for all $\epsilon > 0$ we obtain the result. □

Example (in lieu of 11.4.2). (i) For the entire function sin and any simple closed contour γ with positive orientation and enclosing z_0 we use the Cauchy Integral formula to obtain

$$\oint_\gamma \frac{\sin(z)}{z - z_0}\, dz = 2\pi i \sin(z_0).$$

(ii) For the entire function $f(z) = \exp(z)$ and a circle σ with positive orientation and and radius smaller than one and centered at 0, what is the value of

$$\oint_\sigma \frac{\exp(z)}{z - z^2}\, dz?$$

The integrand here is

$$\frac{\exp(z)}{z - z^2} = \frac{1}{z}\left(\frac{\exp(z)}{1 - z}\right) = \frac{1}{z - 0}\left(\frac{\exp(z)}{1 - z}\right).$$

The function

$$f(z) = \frac{\exp(z)}{1 - z}$$

is holomorphic on a simply connected open set U containing σ but not containing the root of $1 - z$.

Thus we can apply the Cauchy Integral formula, with $z_0 = 0$, to obtain

$$\oint_\sigma \frac{\exp(z)}{z - z^2}\, dz = \oint_\sigma \frac{f(z)}{z - 0}\, dz = 2\pi i f(0) = 2\pi i \left(\frac{\exp(0)}{1 - 0}\right) = 2\pi i.$$

Corollary 11.4.3 (Gauss's Mean Value Theorem). For a simply connected open set U and a holomorphic function $f : U \to X$, if the circle $\sigma = \{z \in \mathbb{C} : |z - z_0| = r\}$ is in U, then for the standard parameterization $\sigma(\theta) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$, there holds

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

Proof. Applying the Cauchy Integral Formula we obtain

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta, \end{aligned}$$

which gives the result. □

Remark 11.4.4. The Mean Value Theorems we have encountered in the past give the existence of a point where the mean value is realized. Gauss's Mean Value Theorem not only gives the existence of the point where the mean value is realized, but specifies what the point is. This is another property that holomorphic functions have, a property not found for real differentiable functions.

11.4.2 Riemann's Theorem and Cauchy's Differentiation Theorem

Extensions of Cauchy's Integral formula to derivatives requires a complex-variable version of Leibniz's Rule of passing differentiation inside an integral. The book does not state this version of Leibniz's Rule, but expects you to deduce the appropriate hypotheses for it. Appropriate hypotheses are presented next.

Remark: Leibniz's Rule for Complex Functions. Let U be a simply connected open subset of $\mathbb{C}$ and γ a contour (not assumed simply closed) that lies entirely in U. For Ω either the interior of γ or the exterior of γ intersected with U, if $F : \gamma \times \Omega \to X$ is a continuous function whose partial complex derivative $D_2 F$ exists and is continuous on Ω, then the function

$$w \to \int_\gamma F(z,w)\ dz$$

is holomorphic on Ω and

$$\frac{d}{dw}\int_\gamma F(z,w)\ dz = \int_\gamma \frac{\partial F}{\partial w}(z,w)\ dz.$$

See the Appendix for a proof of this.

Theorem 11.4.5 (Riemann's Theorem). If f is continuous on a closed contour γ, then for every $n \in \mathbb{N}$, the function

$$F_n(w) = \oint_\gamma \frac{f(z)}{(z-w)^n}\ dz$$

is holomorphic on $\mathbb{C} \setminus \gamma$ and its derivative satisfies

$$F_n'(w) = nF_{n+1}(w).$$

Proof. Let Ω be either the interior or exterior of γ, i.e., one of the two connected components of $\mathbb{C} \setminus \gamma$.

For each $n \in \mathbb{N}$, the hypotheses imply that the function $G_n : \gamma \times \Omega$ defined by

$$G_n(z,w) = \frac{f(z)}{(z-w)^n}$$

is continuous with continuous derivative

$$D_2 G_n(z,w) = \frac{\partial G_n}{\partial w} = \frac{nf(z)}{(z-w)^{n+1}} = nG_{n+1}(z,w)$$

on Ω.

By Liebniz's Rule for complex functions the function $F_n : \Omega \to X$ defined by

$$F_n(w) = \oint_\gamma G_n(z,w)\ dz = \oint_\gamma \frac{f(z)}{(z-w)^n}\ dz$$

is holomorphic on Ω, and we can pass the derivative through the integral:

$$F_n'(w) = \frac{d}{dw}F_n(w) = \oint_\gamma \frac{\partial G_n}{\partial w}(z,w)\ dz = \oint_\gamma nG_{n+1}(z,w)\ dz = nF_{n+1}(w).$$

This gives the result. $\square$

Now we can state and prove an amazing result that connects the derivative of a holomorphic function with the function through a contour integral.

Corollary 11.4.6 (Cauchy's Differentiation Formula). For a simply connected open set U in $\mathbb{C}$, let $f : U \to X$ be holomorphic and let γ be a simple closed contour lying entirely in U, traversed once in the counterclockwise direction. If w is in the interior of γ, then for any $k = 0, 1, 2, \ldots$, there holds

$$f^{(k)}(w) = \frac{k!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-w)^{k+1}}\ dz.$$

Proof. We proceed by induction.

The base case of $k = 0$ is Cauchy's Integral formula:

$$f^{(0)}(w) = f(w) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-w}\ dz = \frac{0!}{2\pi i}\oint \frac{f(z)}{(z-w)^{0+1}}\ dz.$$

Now for some $k = 0, 1, 2, \ldots$, suppose that

$$f^{(k-1)}(w) = \frac{(k-1)!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-w)^k}\ dz.$$

By Riemann's Theorem (Theorem 11.4.5), the function $f^{(k-1)}$ is holomorphic on the interior of γ and for each w in the interior of γ there holds

$$f^k(w) = \frac{d}{dw}f^{(k-1)}(w) = \frac{k(k-1)!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-w)^{k+1}}\ dz = \frac{k!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-w)^{k+1}}\ dz.$$

This gives the result. $\square$

Nota Bene 11.4.7. Often, we will use Cauchy's Differentiation formula in the form

$$\oint_\gamma \frac{f(z)}{(z-w)^k}\ dz = \frac{2\pi i}{(k-1)!}\, f^{(k-1)}(w)$$

for w in the interior of γ. See Example 11.4.10.

Corollary 11.4.8. For a simply connected open set U in $\mathbb{C}$, if $f : U \to X$ is holomorphic, then f' is holomorphic on U; hence by induction f is infinitely holomorphic, i.e., for all $k = 1, 2, 3, \ldots$, the complex derivative f^k exists on U.

Remark 11.4.9. That a holomorphic function is infinitely holomorphic is quite remarkable; a real differentiable function is not guaranteed to be twice differentiable.

Example 11.4.10. We demonstrate how to use Cauchy's Differentiation formula (and Nota Bene 11.4.7) to compute certain types of contour integrals.

(i) For γ a simple closed curve enclosing 0, compute

$$\oint_\gamma \frac{\cos(z)}{2z^3}\ dz.$$

By setting $f(z) = \cos(z)/2$ we recognize the contour integral as

$$\oint_\gamma \frac{f(z)}{(z-0)^3}\ dz.$$

By Nota Bene 11.4.7 we have

$$\oint_\gamma \frac{f(z)}{(z-0)^3}\ dz = \frac{2\pi i}{(3-1)!}\, f^{(3-1)}(0) = \pi i f^{(2)}(0).$$

Since

$$f'(z) = -\frac{\sin(z)}{2},\ f^{(2)}(z) = -\frac{\cos(z)}{2},$$

we compute

$$\oint_\gamma \frac{\cos(z)}{2z^3}\ dz = -\pi i\frac{\cos(0)}{2} = -\frac{\pi i}{2}.$$

(In lieu of (ii)) For γ the circle of radius 1 centered at i, compute

$$\oint_\gamma \frac{(2+z)e^z}{(z^2-1)(z-i)^2}\ dz.$$

By setting $f(z) = (2+z)e^z/(z^2-1)$ we recognize the contour integral as

$$\oint_\gamma \frac{f(z)}{(z-i)^2}\ dz.$$

The two roots ± 1 of $z^2 - 1$ are located outside of γ, so that $f(z)$ is holomorphic on a simply connected open set U containing γ.

By Nota Bene 11.4.7 we have

$$\oint_\gamma \frac{f(z)}{(z-i)^2}\ dz = \frac{2\pi i}{(2-1)!}\, f^{(2-1)}(i) = 2\pi i f'(i).$$

Since

$$\begin{aligned} f'(z) &= \frac{(e^z + (2+z)e^z)(z^2-1) - (2+z)e^z(2z)}{(z^2-1)^2} \\ &= \frac{e^z[z^3 + z^2 - 5z - 3]}{(z^2-1)^2}, \end{aligned}$$

we compute

$$\oint_\gamma \frac{(2+z)e^z}{(z^2-1)(z-i)^2}\ dz = 2\pi i\frac{e^i[i^3+i^2-5i-3]}{(i^2-1)^2} = 2\pi i\frac{e^i[-6i-4]}{4} = \pi e^i(3-2i).$$

(In lieu of (iii)) For γ the circle of radius 2 centered at 0, compute

$$\oint_\gamma \frac{\sin(z)}{(z+1)^2(z-1)}\ dz.$$

We cannot directly apply Cauchy's Integral formulas to this contour integral because the integrand fails to be complex differentiable at the two points -1 and 1 lying inside γ.

To overcome this we form two small circles γ_{-1} around -1 and γ_1 around 1, use a cut σ to connect γ to γ_1, and use a cut τ to connect γ_{-1} to γ_1. (See Figure 11.2 on page 429 in book.)

A simply connected set is formed by the curve

$$\gamma + \sigma - (\text{bottom half of } \gamma_1) + \tau - \gamma_{-1} - \tau - (\text{top half of } \gamma_1) - \sigma.$$

(The book has incorrect orientations on the two small simple closed curves.)

The Cauchy-Goursat Theorem guarantees that

$$\oint_\gamma \frac{\sin(z)}{(z+1)^2(z-1)}\ dz = \oint_{\gamma_{-1}} \frac{\sin(z)}{(z+1)^2(z-1)}\ dz + \oint_{\gamma_1} \frac{\sin(z)}{(z+1)^2(z-1)}\ dz,$$

as the contour integrals over σ and τ cancel.

By setting

$$f_{-1} = \frac{\sin(z)}{z-1} \text{ and } f_1(z) = \frac{\sin(z)}{(z+1)^2}$$

we recognize that

$$\oint_{\gamma_{-1}} \frac{\sin(z)}{(z+1)^2(z-1)}\ dz = \oint_{\gamma_{-1}} \frac{f_{-1}(z)}{(z+1)^2}\ dz$$

and

$$\oint_{\gamma_1} \frac{\sin(z)}{(z+1)^2(z-1)}\ dz = \oint_{\gamma_1} \frac{f_1(z)}{z-1}\ dz.$$

By Nota Bene 11.4.7 we have

$$\oint_{\gamma_{-1}} \frac{f_{-1}(z)}{(z+1)^2}\ dz = 2\pi i f_{-1}'(-1) \text{ and } \oint_{\gamma_1} \frac{f_1(z)}{z-1}\ dz = 2\pi i f_1(1).$$

Since

$$f_{-1}'(z) = \frac{(z-1)\cos(z) - \sin(z)}{(z-1)^2}$$

we obtain

$$2\pi i f_{-1}'(-1) = 2\pi i\left(\frac{-2\cos(-1) - \sin(-1)}{4}\right) \text{ and } 2\pi i f_1(1) = 2\pi i\left(\frac{\sin(1)}{4}\right).$$

Thus

$$\oint_\gamma \frac{\sin(z)}{(z+1)^2(z-1)}\ dz = 2\pi i\left(-\frac{\cos(1)}{2} + \frac{\sin(1)}{2}\right) = \pi i(-\cos(1)+\sin(1)).$$

Appendix

Proof of Leibniz's Rule for complex variable functions.

We are going to make use of a result that relates the complex derivative with the real derivatives: for $w = \xi + i\eta$, if $f(w) = u(\xi, \eta) + iv(\xi, \eta)$, then

$$\frac{d}{dw} f(w) = \frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right)\big(u(\xi, \eta) + iv(\xi, \eta)\big).$$

We will see a proof of this in a sightly different context below.

Now WLOG suppose that γ is a smooth curve with parameterization $\gamma : [a, b] \to \mathbb{C}$; we have $z = \gamma(t)$.

The variable $w \in \Omega$ can be written in terms of its real and imaginary parts

$$w = \xi + i\eta.$$

Thus image of the function $F(z, w)$ can be written in terms of its real and imaginary parts in the complex Banach space X, i.e.,

$$F(z, w) = u(z, \xi, \eta) + iv(z, \xi, \eta).$$

The hypothesis of F being continuous on $\gamma \times \Omega$ implies that u and v are continuous functions on $\gamma \times \Omega$ where we now think of Ω as an open set in $\mathbb{R}^2$.

The partial complex derivative $D_2F = \partial F/\partial w$ can be expressed in terms of the real variables ξ and η by

$$\frac{\partial F}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right)\big(u(z, \xi, \eta) + iv(z, \xi, \eta)\big).$$

This is because (i.e., the proof) the existence of $\partial F/\partial w$ on Ω implies that for $w_0 = \xi_0 + i\eta_0$ and $\xi + i\eta_0 \to w_0$, i.e., $\xi \to \xi_0$, we have

$$\frac{\partial F}{\partial w}(w_0) = \lim_{w \to w_0} \frac{F(z, w) - F(z, w_0)}{w - w_0} = \lim_{\xi \to \xi_0} \frac{F(z, \xi, \eta_0) - F(z, \xi_0, \eta_0)}{\xi - \xi_0} = \frac{\partial F}{\partial \xi}(z, \xi_0, \eta_0),$$

(see proof of Theorem 11.1.4) which implies that

$$\frac{\partial F}{\partial w} = \frac{\partial u}{\partial \xi} + i\frac{\partial v}{\partial \xi},$$

while, from the Cauchy-Riemann equations, we have

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right)\big(u(t, \xi, \eta) + iv(t, \xi, \eta)\big) \\
&\quad = \frac{1}{2}\left(\frac{\partial u}{\partial \xi} + i\frac{\partial v}{\partial \xi} - i\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \eta}\right) \\
&\quad = \frac{1}{2}\left(\frac{\partial u}{\partial \xi} + i\frac{\partial v}{\partial \xi} + i\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \xi}\right) \\
&\quad = \frac{\partial u}{\partial \xi} + i\frac{\partial v}{\partial \xi}.
\end{aligned}$$

Similarly one obtains (by letting $\xi_0 + i\eta \to w_0$, i.e., $\eta \to \eta_0$) that

$$\frac{\partial F}{\partial w} = -i\frac{\partial F}{\partial \eta} = -i\left(\frac{\partial u}{\partial \eta} + i\frac{\partial v}{\partial \eta}\right) = -i\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \eta},$$

which is equal to

$$\frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right)\big(u(z,\xi,\eta) + iv(z,\xi,\eta)\big) = -i\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \eta}$$

by the Cauchy-Riemann equations.

The hypothesis of D_2F being continuous on Ω implies that the real partial derivatives $\partial u/\partial\xi$, $\partial u/\partial\eta$, $\partial v/\partial\xi$, and $\partial v/\partial\eta$ are continuous on Ω.

We can now apply Leibniz's Rule to obtain the real C^1 differentiability of

$$\psi(\xi,\eta) = \int_a^b F(\gamma(t),\xi,\eta)\gamma'(t)\ dt = \int_a^b \big(u(\gamma(t),\xi,\eta) + iv(\gamma(t),\xi,\eta)\big)\gamma'(t)\ dt.$$

Both partial derivatives of ψ exist by Liebniz's Rule applied with $\partial/\partial\xi$ and then with $\partial/\partial\eta$.

These partial derivatives are each continuous on Ω; the continuity of the partial derivatives is not stated in book's version of Leibniz's Rule, but in the proof given in Lecture Note #17, they were shown to be continuous; hence ψ is C^1 on Ω.

Furthermore, by Liebniz's Rule, the partial derivatives of ψ are

$$\frac{\partial \psi}{\partial \xi} = \int_a^b \frac{\partial F}{\partial \xi}(\gamma(t),\xi,\eta)\gamma'(t)\ dt = \int_\gamma \frac{\partial F}{\partial w}(z,w)\ dz$$

and

$$\frac{\partial \psi}{\partial \eta} = \int_a^b \frac{\partial F}{\partial \eta}(\gamma(t),\xi,\eta)\gamma'(t)\ dt = i\int_\gamma \frac{\partial F}{\partial w}(z,w)\ dz.$$

These imply the Cauchy-Riemann equation for ψ, i.e.,

$$\frac{\partial \psi}{\partial \eta} = i\frac{\partial \psi}{\partial \xi}.$$

Thus ψ is holomorphic on Ω with

$$\frac{d\psi}{dw} = \frac{\partial \psi}{\partial \xi}.$$

(see the proof of Theorem 11.1.4). Therefore

$$\frac{d}{dw}\int_\gamma F(z,w)\ dz = \int_\gamma \frac{\partial F}{\partial w}(z,w)\ dz$$

and this completes the proof. $\square$