Math 346 Lecture #28
11.5 Consequences of Cauchy's Integral Formula

We saw in Section 11.4 some consequences of Cauchy's Integral formula, namely Gauss's Mean Value Theorem and Cauchy's Differentiation formula. We look at more consequences of Cauchy's Integral formula in this lecture. Some of these consequences only hold for complex-valued holomorphic functions and not the more general complex Banach-space value holomorphic functions. We will point which is the case for each consequence.

As always, we let $(X, \|\cdot\|_X)$ be a complex Banach space.

11.5.1 Liouville's Theorem

The following consequence of Cauchy's Differentiation formula holds for complex Banach space valued holomorphic functions.

Theorem 11.5.1 (Liouville's Theorem). If $f : \mathbb{C} \to X$ is entire and bounded, i.e., there is $M > 0$ such that $\|f(z)\|_X \leq M$ for all $z \in \mathbb{C}$, then f is a constant function.

Proof. Fix $z_0 \in \mathbb{C}$ and the circle γ in $\mathbb{C}$ with center z_0 and radius $R > 0$.

We use the standard parameterization of γ, namely $\gamma(\theta) = z_0 + Re^{i\theta}$, $\theta \in [0, 2\pi]$.

By the assumed boundedness of f and Cauchy's Differentiation formula we have

$$\begin{aligned}
\|f'(z_0)\|_X &= \left\| \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^2}\, dz \right\|_X \\
&\leq \frac{1}{2\pi} \oint_\gamma \frac{\|f(z)\|_X}{|z - z_0|^2}\, |dz| \\
&\leq \frac{1}{2\pi} \oint_\gamma \frac{M}{|z - z_0|^2}\, |dz| \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{|Re^{i\theta}|^2}\, |Rie^{i\theta}| d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{R}\, d\theta \\
&= \frac{M}{R}.
\end{aligned}$$

This inequality holds for all $R > 0$ implying that $\|f'(z_0)\|_X = 0$.

The arbitrariness of $z_0 \in \mathbb{C}$ implies that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$.

By Proposition 11.2.4, the entire function f is constant. □

Example 11.5.2. The nonconstant entire functions $\cos(z)$ and $\sin(z)$ are bounded when $z \in \mathbb{R}$, but by Liouville's Theorem are not bounded on $\mathbb{C}$. You have HW (Exercise 11.20) to find sequences $\{z_n\}$ and $\{w_n\}$ in $\mathbb{C}$ for which $|\sin(z_n)| \to \infty$ and $|\cos(w_n)| \to \infty$. You are given a hint for $\sin(z)$, but here are some better hints: for $z = x + iy$, there holds

$$\sin(z) = \sin x \cosh y + i \cos x \sinh y, \quad \cos(z) = \cos x \cosh y - i \sin x \sinh y.$$

Example. A complex Banach space is the complex vector space $M_n(\mathbb{C})$ equipped with the induced matrix norm $\|\cdot\|_\infty$. The function $f:\mathbb{C}\to M_2(\mathbb{C})$ defined by

$$f(z)=\begin{bmatrix}1 & 0\\ 0 & z\end{bmatrix}$$

is entire because for any $z_0\in\mathbb{C}$ we have

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{z\to z_0}\frac{1}{z-z_0}\begin{bmatrix}0 & 0\\ 0 & z-z_0\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

The entire function f is not constant, and so by the contrapositive of Liouville's Theorem its norm is not bounded; explicitly we have

$$\|f(z)\|_\infty=\max\{1,|z|\}\to\infty$$

as $|z|\to\infty$.

This matrix valued function f is readily generalized to $n\geq 3$.

This next result only holds for complex-valued entire functions.

Corollary 11.5.3. If $f:\mathbb{C}\to\mathbb{C}$ is entire and there exists $\epsilon>0$ such that $|f(z)|\geq\epsilon$ for all $z\in\mathbb{C}$, i.e., uniformly bounded away from zero, then f is constant.

Proof. The assumptions of $f:\mathbb{C}\to\mathbb{C}$ and $|f(z)|\geq\epsilon$ for all $z\in\mathbb{C}$ imply that

$$\left|\frac{1}{f(z)}\right|\leq\frac{1}{\epsilon}$$

for all $z\in\mathbb{C}$.

Since f is entire and bounded away from zero, the function $1/f$ is entire.

By Liouville's Theorem, with $M=1/\epsilon$, the function $1/f$ is a constant function, i.e., there is $c\in\mathbb{C}$ such that $1/f(z)=c$ for all $z\in\mathbb{C}$.

The constant c satisfies $1=f(z)c$ for all $z\in\mathbb{C}$, implying that $c\neq 0$.

Thus $f(z)=1/c$ for all $z\in\mathbb{C}$. □

11.5.2 The Fundamental Theorem of Algebra

We use Corollary 11.5.3 to show that a nonconstant polynomial function f from $\mathbb{C}$ to $\mathbb{C}$ has at least one root, i.e., there is $z_0\in\mathbb{C}$ such that $f(z_0)=0$.

Theorem 11.5. (Fundamental Theorem of Algebra). Every nonconstant polynomial function from $\mathbb{C}$ to $\mathbb{C}$ has at least one root in $\mathbb{C}$.

Proof. Let $f:\mathbb{C}\to\mathbb{C}$ be nonconstant polynomial of degree $k\geq 1$: there are coefficients $b_0,b_1,\dots,b_{k-1},b_k\in\mathbb{C}$ with $b_k\neq 0$ such that

$$f(z)=b_kz^k+b_{k-1}z^{k-1}+\cdots+b_1z+b_0.$$

Define the nonconstant polynomial $p:\mathbb{C}\to\mathbb{C}$ by

$$p(z)=z^k+a_{k-1}z^{k-1}+\cdots+a_1z+a_0$$

where

$$a_j = \frac{b_j}{b_k}, \ j = 0, 1, \ldots, k-1.$$

The polynomial p is a monic polynomial with the same roots (if any) as f.

Set $a = \max\{|a_{k-1}|, \ldots, |a_1|, |a_0|\}$.

There are two cases to consider: $a = 0$ and $a > 0$.

In the case of $a = 0$, we have $p(z) = z^k$ and this polynomial has $z_0 = 0$ as a root.

For the case of $a > 0$ suppose that p does not have any roots.

By the "reverse" triangle inequality $|c| - |d| \leq |c - d|$ applied to $c = z^k$ and $d = -(a_{k-1}z^{k-1} + \cdots + a_1 z + a_0)$ we obtain

$$|p(z)| = |c - d| \geq |c| - |d| = |z^k| - |a_{k-1}z^{k-1} + \cdots + a_1 z + a_0|.$$

By repeated use of the triangle inequality we have

$$\begin{aligned}
|a_{k-1}z^{k-1} + \cdots + a_1 z + a_0| &\leq |a_{k-1}z^{k-1}| + \cdots + |a_1 z| + |a_0| \\
&= |a_{k-1}|\,|z|^{k-1} + \cdots + |a_1|\,|z| + |a_0| \\
&\leq a|z|^{k-1} + \cdots + a|z| + a \\
&= ka(|z|^{k-1} + \cdots + |z| + 1).
\end{aligned}$$

Thus

$$-|a_{k-1}z^{k-1} + \cdots + a_1 z + a_0| \geq -ka(|z|^{k-1} + \cdots + |z| + 1).$$

Set $R = \max\{(k+1)a, 1\}$.

Then $R \geq 1$ and $R \geq (k+1)a$, the latter implying that $R - ka \geq a$.

Because $R \geq 1$ we have for all $j = 0, 1, 2, \ldots, k-2$ that

$$|z|^j \leq |z|^{k-1},$$

whence for all $j = 0, 1, 2, \ldots, k-2$ that

$$-|z|^j \geq -|z|^{k-1}.$$

Thus for $|z| \geq R$ there holds

$$\begin{aligned}
|p(z)| &\geq |z|^k - \big(|a_{k-1}|\,|z|^{k-1} + \cdots + |a_1|\,|z| + |a_0|\big) \\
&\geq |z|^k - a(|z|^{k-1} + \cdots + |z| + 1) \\
&\geq |z|^k - a(|z|^{k-1} + \cdots + |z|^{k-1} + |z|^{k-1}) \\
&= |z|^k - ka|z|^{k-1} \\
&= |z|^{k-1}(|z| - ka) \\
&\geq |z| - ka \\
&\geq R - ka \\
&\geq a.
\end{aligned}$$

This says that p has no roots on the set $\{z \in \mathbb{C} : |z| \geq R\}$ and that p is uniformly bounded away from 0 on this set.

Our assumption that p has no roots means that the continuous function $z \to |p(z)|$ on the compact set $\{z \in \mathbb{C} : |z| \leq R\}$ is bounded away from zero on this set.

Thus there exists $\epsilon > 0$ such that $|p(z)| \geq \epsilon$ for all $z \in \mathbb{C}$.

By Corollary 11.5.3. we conclude that $p(z)$ is a constant function, which is a contradiction.

Therefore the nonconstant function p has a root. $\square$.

Remark. The Fundamental Theorem of Algebra is an existence result – its proof does not give an algorithm for finding the roots. You have it as HW (Exercise 11.21) to show that a polynomial $p_n(z)$ of degree n has exactly n roots (counting multiple roots). Hint: use the Fundamental Theorem of Algebra to find a root, say z_n of $p_n(z)$, then form a new polynomial $p_{n-1}(z)$ of degree $n-1$ obtained by dividing $p_n(z)$ by the factor $z - z_n$. Is there a root z_{n-1} of p_{n-1}?

11.5.3 The Maximum Modulus Principle

For an open set U in $\mathbb{C}$ and a holomorphic function $f : U \to \mathbb{C}$, the continuous function $z \to |f(z)|$, on any compact subset K of U, attains its maximum value at some point of K by the Extreme Value Theorem. When the open interior of K is a nonempty and path-connected, the Maximum Modulus Principle states that the point where the maximum of $|f|$ is attained must be on the boundary of K.

The Maximum Modulus Principle is a consequence of the following results that apply to general complex Banach spaced valued holomorphic functions. The book states Lemma 11.5.6 only for complex-valued holomorphic functions, but its proof works for general complex Banach spaced value holomorphic functions.

Lemma 11.5.6. For an open U in $\mathbb{C}$ and $f : U \to X$ holomorphic, if $\|f\|_X$ attains its supremum at $z_0 \in U$, then $\|f\|_X$ is constant in every open ball $B(z_0, r)$ whose closure $\overline{B(z_0, r)}$ is contained in U.

Proof. Let $r > 0$ be such that $\overline{B(z_0, r)} \subset U$.

By Gauss's Mean Value Theorem (Corollary 11.4.3), with the standard parameterization of the boundary of $\overline{B(z_0, r)}$, we have

$$\|f(z_0)\|_X = \frac{1}{2\pi}\left\| \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta \right\|_X \leq \frac{1}{2\pi}\int_0^{2\pi} \|f(z_0 + re^{i\theta})\|_X\, d\theta.$$

By hypothesis, $\|f(z_0 + re^{i\theta})\|_X \leq \|f(z_0)\|_X$ for all $\theta \in [0, 2\pi]$, and this implies that

$$\frac{1}{2\pi}\int_0^{2\pi} \|f(z_0 + re^{i\theta})\|_X\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \|f(z_0)\|_X\, dt = \|f(z_0)\|_X.$$

Thus

$$\|f(z_0)\|_X \leq \frac{1}{2\pi}\int_0^{2\pi} \|f(z_0 + re^{i\theta})\|_X\, d\theta \leq \|f(z_0)\|_X$$

which implies that

$$\|f(z_0)\|_X = \frac{1}{2\pi}\int_0^{2\pi} \|f(z_0 + re^{i\theta})\|_X \, d\theta.$$

From this follows

$$0 = \|f(z_0)\|_X - \frac{1}{2\pi}\int_0^{2\pi} \|f(z_0 + re^{i\theta})\|_X \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \big(\|f(z_0)\|_X - \|f(z_0 + re^{i\theta})\|_X\big) \, d\theta.$$

Since $\|f(z_0)\|_X \geq \|f(z_0 + re^{i\theta})\|_X$ for all $\theta \in [0, 2\pi]$, the continuous integrand satisfies

$$\|f(z_0)\|_X - \|f(z_0 + re^{i\theta})\|_X \geq 0.$$

The integral of a nonnegative continuous function being zero implies that the integrand is the zero function (see the HW problem 8.6).

Thus $\|f(z_0 + re^{i\theta})\|_X = \|f(z_0)\|_X$ for all $\theta \in [0, 2\pi]$.

The same argument shows that for any $0 < \epsilon < r$ there holds $\|f(z_0 + \epsilon e^{i\theta})\|_X = \|f(z_0)\|_X$ for all $\theta \in [0, 2\pi]$.

This implies that $\|f\|_X$ is constant on $B(z_0, r)$. □

Lemma (Precursor to Maximum Modulus Theorem). For U an open, path-connected subset of $\mathbb{C}$ and $f : U \to X$ holomorphic, if $\|f\|_X$ is not constant on U, then the continuous function $z \to \|f(z)\|_X$ never attains its supremum on U.

Proof. We proceed by way of contradiction: suppose that $\|f\|_X$ is not constant on U but that $\|f\|_X$ attains its supremum at some point $z_0 \in U$.

Since $\|f\|_X$ is non constant and $\|f(z_0)\|_X$ is the maximum of $\|f\|_X$ on U, there exists $w \in U \setminus \{z_0\}$ such that $\|f(w)\|_X < \|f(z_0)\|_X$.

By the path-connectedness of U, there is a contour $\gamma : [a, b] \to U$ with $\gamma(a) = z_0$ and $\gamma(b) = w$.

Since γ is compact and U^c is closed, the quantity

$$\epsilon = d(\gamma, U^c) = \inf\{|c - d| : c \in \gamma, d \in U^c\}$$

is positive (see Exercise 5.33).

An open cover of γ is the collection of open balls

$$\{B(\gamma(t), \epsilon) : t \in [a, b]\}$$

where for each $\gamma(t)$ there holds $B(\gamma(t), \epsilon/2) \subset U$ by the definition of ϵ.

By compactness of γ there is a finite subcover of these open balls.

WLOG we choose finitely open balls $B(\gamma(t_j), \epsilon)$, $j = 0, \ldots, n$, where the values of t_j satisfy

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,$$

and, writing $z_j = \gamma(t_j)$ for $j = 0, 1, \ldots, n-1$ and $w = \gamma(t_n)$, that

$$z_j, z_{j+1} \in B(z_j, \epsilon) \cap B(z_{j+1}, \epsilon) \text{ for all } j = 0, \ldots, n-1.$$

[This last condition means that the arclength between consequent points z_j, z_{j+1} on γ is smaller than $\epsilon/2$.]

Because $\|f(z_0)\|_X$ is the maximum of $\|f\|_X$ on U, we have by Lemma 11.5.6 that the function $\|f\|_X$ is constant on $B(z_0, \epsilon/2)$.

Since $z_1 \in B(z_0, \epsilon/2)$ we have that $\|f(z_1)\|_X = \|f(z_0)\|_X$.

Again by Lemma 11.5.6, the function $\|f\|_X$ is constant on $B(z_1, \epsilon/2)$, and this implies since $z_2 \in B(z_1, \epsilon/2)$ that $\|f(z_2)\|_X = \|f(z_1)\|_X = \|f(z_0)\|_X$.

Continuing this process leads to having $\|f(z_{n-1})\|_X = \|f(z_0)\|_X$ and that $\|f\|_X$ is constant on $B(z_{n-1}, \epsilon/2)$.

Since $w \in B(z_{n-1}, \epsilon/2)$, we have the contradiction $\|f(w)\|_X = \|f(z_0)\|_X$. □

Example. We consider again the matrix valued entire function

$$f(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \in M_2(\mathbb{C}).$$

For the open, path-connected set $U = B(0,2)$ in $\mathbb{C}$, the restriction $f : U \to M_2(\mathbb{C})$ is holomorphic and in terms of the induced matrix norm $\|\cdot\|_\infty$, we have

$$\|f(z)\|_\infty = \begin{cases} 1 & \text{if } z \in \overline{B(0,1)}, \\ |z| & \text{if } z \in B(0,2) \setminus \overline{B(0,1)}. \end{cases}$$

The nonconstant function $\|f\|_\infty$ does not attain is supremum of 2 on U.

To get the Maximum Modulus Principle as stated at the beginning of this subsection, we restrict to complex valued holomorphic functions f. We do this so we can use Proposition 11.1.7 which states that if $|f|$ is constant on an open, path-connected set, then f is constant. For general complex Banach spaces it is not true that $\|f\|_X$ is constant on an open, path-connected set implies f is constant, as illustrated next.

Example. We consider again the matrix valued entire function

$$f(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} \in M_2(\mathbb{C}).$$

For the open, path-connected set $U = B(0,1)$ in $\mathbb{C}$, the restriction $f : U \to M_2(\mathbb{C})$ is a nonconstant holomorphic function.

But in terms of the induced matrix norm $\|\cdot\|_\infty$, we have $\|f(z)\|_\infty = 1$ for all $z \in U$.

Theorem 11.5.5 (The Maximum Modulus Principle). For an open, path-connected U in $\mathbb{C}$ and a holomorphic $f : U \to \mathbb{C}$, if f is not constant on U, then $|f|$ does not attain its supremum on U.

Proof. We proceed by way of the contrapositive: suppose $|f|$ attains its supremum on U.

By the contrapositive of the Precursor to the Maximum Modulus Principle, the continuous function $|f|$ is constant.

By Proposition 11.1.7, the function f is constant on U. □

The next result, a corollary of the Maximum Modulus Principle, is stated in an imprecise manner in the book. Here is a precise version.

Corollary 11.5.7. For a compact set D whose interior D° is nonempty and path-connected, if $f : D \to \mathbb{C}$ is continuous and holomorphic on D°, then $|f|$ attains is maximum on the boundary of D.

Proof. Continuity of function $|f| : D \to \mathbb{R}$ on the compact D implies that $|f|$ attains its maximum at some point in D.

If f a constant function, then $|f|$ is a constant function, and the maximum of $|f|$ is attained at every point of D, including all boundary points of D.

If f is not a constant function, then the Maximum Modulus Principle implies that the maximum of $|f|$ cannot be achieved on D°, hence that the maximum of $|f|$ is achieved on the boundary of D. □