Math 346 Lecture #29
11.6 Power Series and Laurent Series

We have already seen that a convergent power series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ gives a holomorphic function on $B(z_0, r)$ for some $r > 0$. In this lecture we see the converse of this: every function f holomorphic on an open set U can be written as a convergent power series on $B(z_0, r) \subset U$ for every $z_0 \in U$ and some $r > 0$. We will also see that there is an extension of power series, called Laurent series, that gives a way to write a function that is holomorphic on a punctured ball or disk $B(z_0, r) \setminus \{z_0\}$, that describes the kind of singularity the function has at z_0.

Throughout we assume that $(X, \|\cdot\|_X)$ is a complex Banach space.

11.6.1 Power Series

For a function $f : U \to X$ holomorphic on an open U in $\mathbb{C}$, we can form for each $z_0 \in U$ the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

because f is infinitely holomorphic on U by Corollary 11.4.8. Using Cauchy's Integral and Differentiation formulas we show that the Taylor series of f at z_0 converges to f on some open ball $B(z_0, r) \subset U$ for $r > 0$.

Theorem 11.6.1. For an open U in $\mathbb{C}$, if $f : U \to X$ is holomorphic, then for each $z_0 \in U$ there exists a largest $r \in (0, \infty]$ such that $B(z_0, r) \subset U$ and the Taylor series for f at z_0 converges uniformly to f on compact subset of $B(z_0, r)$.

Proof. For a fixed $z_0 \in U$, there is a largest $r > 0$ by the openness of U such that $B(z_0, r) \subset U$.

For any $0 < \epsilon < r$ consider the compact set $D = \overline{B(z_0, r-\epsilon)} \subset B(z_0, r)$.

To get uniform convergence of the Taylor series on compact subset of $B(z_0, r)$ is suffices to show uniform convergence on sets of the form D.

The circle γ given by

$$\{w \in \mathbb{C} : |w - z_0| = r - \epsilon/2\}$$

lies in $B(z_0, r) \setminus D$, so that γ encloses D.

We orient γ with the positive orientation.

For any $z \in D$ and any $w \in \gamma$ we have $|z - z_0| \le r - \epsilon$ and $|w - z_0| = r - \epsilon/2$; thus

$$\left|\frac{z-z_0}{w-z_0}\right| = \frac{|z-z_0|}{|w-z_0|} \le \frac{r-\epsilon}{r-\epsilon/2} < 1.$$

This implies convergence of the geometric series in $(z - z_0)/(w - z_0)$,

$$\frac{1}{\left(1 - \frac{z-z_0}{w-z_0}\right)} = \sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k = \sum_{k=0}^{\infty}\frac{1}{(w-z_0)^k}(z-z_0)^k.$$

This convergence is uniform and absolute on D for each $w \in \gamma$ by the Lemma 11.2.5 (Abel-Weierstrass Lemma) because for $a_k = 1/(w-z_0)^k$, $R = |w - z_0|$, and $M = 1$ there holds

$$|a_k|R^k = \frac{1}{|w-z_0|^k}|w-z_0|^k = 1 \leq M.$$

[We get $|a_k| = 1/|w-z_0|^k$ by applying the norm to $(w-z_0)^k a_k = 1$.]

Since

$$(w-z_0)\left(1 - \frac{z-z_0}{w-z_0}\right) = w - z_0 - (z-z_0) = w - z$$

we obtain

$$\frac{1}{w-z} = \frac{1}{w-z_0}\frac{1}{\left(1-\frac{z-z_0}{w-z_0}\right)} = \frac{1}{w-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k = \sum_{k=0}^{\infty}\frac{(z-z_0)^k}{(w-z_0)^{k+1}}.$$

Since γ is compact, the restriction of the continuous f to γ is bounded; this implies that

$$\frac{f(w)}{w-z} = \sum_{k=0}^{\infty}\frac{f(w)(z-z_0)^k}{(w-z_0)^{k+1}}$$

is also uniformly and absolutely convergent on D for each $w \in \gamma$.

By the Cauchy Integral formula we have that

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{\gamma}\left(\sum_{k=0}^{\infty}\frac{f(w)(z-z_0)^k}{(w-z_0)^{k+1}}\right)\,dw.$$

Since integration is a continuous linear transformation, it commutes with uniform limits, which means in particular that integration commutes with uniformly convergent sums:

$$f(z) = \sum_{k=0}^{\infty}\frac{1}{2\pi i}\oint_{\gamma}\frac{f(w)(z-z_0)^k}{(w-z_0)^{k+1}}\,dw = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{\gamma}\frac{f(w)}{(w-z_0)^{k+1}}\,dw\right)(z-z_0)^k.$$

By Cauchy's Differentiation formula we have

$$\frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i}\oint_{\gamma}\frac{f(w)}{(w-z_0)^{k+1}}\,dw.$$

This gives

$$f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

for all $z \in D$, i.e., we have uniformly and absolute convergence of the Taylor series for f at z_0 to f on compact subsets of $B(z_0, r)$. □

Remark 11.6.2. We now have shown that a function $f : U \to X$ is holomorphic on U if and only if f is analytic on U. Because of this equivalence, we often use holomorphic and analytic interchangeably.

Proposition 11.6.3. For $\{a_k\}_{k=0}^{\infty} \subset X$, a convergent power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

about $z_0 \in \mathbb{C}$ is unique and equal to its Taylor series.

Proof. By Theorem 11.2.8, the convergent power series is differentiable with derivative given by the term-by-term derivative of the power series, i.e.,

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^k$$

with the same radius of convergence R as the original power series.

By induction this shows that the j^{th} derivative of f is given by the power series

$$f^{(j)}(z) = \sum_{k=j}^{\infty} k(k-1)\cdots(k-j+1) a_k (z - z_0)^{k-j}$$

with the same radius of convergence R as the original power series.

Thus we obtain for every $j = 0, 1, 2, \ldots,$ that

$$f^{(j)}(z_0) = j! a_j \text{ or } a_j = \frac{f^{(j)}(z_0)}{j!}.$$

This implies that f is equal to its Taylor series at z_0, i.e.,

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j$$

for all $z \in B(z_0, R)$.

Since there is only one Taylor series of f about z_0, we obtain the uniqueness of the power series of f about z_0. $\square$

Corollary 11.6.4. For an open, path-connected subset U of $\mathbb{C}$, and a holomorphic $f : U \to X$, if there exists $z_0 \in U$ such that $f^{(n)}(z_0) = 0$ for all $n = 0, 1, 2, 3, \ldots$, then $f(z) = 0$ for all $z \in U$.

Proof. By way of contradiction, assume the hypothesis and the existence of $w \in U$ such that $f(w) \neq 0$.

The Taylor series of f about z_0 is the zero function because

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k = \sum_{k=0}^{\infty} 0 (z - z_0)^k = 0$$

for all $z \in B(z_0, R)$ where R is the largest $R > 0$ such that $B(z_0, R) \subset U$.

By the path-connectedness of U there is a path $\gamma : [a, b] \to U$ such that $\gamma(a) = z_0$ and $\gamma(b) = w$.

Following the same argument found in the proof of the Precursor of the Maximum Modulus Principle, for $\epsilon = d(\gamma, U^c) > 0$ there is a finite collection of points $z_0, z_1, \ldots, z_n = w$ ordered along the path γ such that $d(z_{j-1}, z_j) < \epsilon/2$ for all $j = 1, 2, \ldots, n$, each ball $B(z_i, \epsilon) \subset U$, and $z_j \in B(z_{j+1}, \epsilon)$ for all $j = 0, 1, \ldots, n-1$.

By Theorem 11.6.4, the holomorphic function has a convergent power series about each z_j, $j = 0, 1, 2, \ldots, n$ with radius of convergence at least as big as $\epsilon > 0$ (the distance from z_j to U^c).

Since $z_1 \in B(z_0, \epsilon)$ and $f(z) = 0$ for all $z \in B(z_0, \epsilon)$, then $f^{(k)}(z_1) = 0$ for all $k = 0, 1, 2, \ldots$.

By Proposition 11.6.3, there holds $f(z) = 0$ for all $z \in B(z_1, \epsilon)$.

Continuing this argument we arrive at $f(z) = 0$ for all $z \in B(w, \epsilon)$, which contradicts $f(w) \neq 0$. $\square$

11.6.2 Zeros of Analytic Functions

Another property of a not identically equal to zero holomorphic function $f : U \to X$ is that its zeros, if any, must be isolated, i.e., if $f(z_0) = 0$, then there exists $\epsilon > 0$ such that $f(z) \neq 0$ for all $z \in B(z_0, \epsilon) \setminus \{z_0\}$. We will use the order of a zero (defined next) to obtain this isolation of zeros.

Definition 11.6.5. For an open U, we say that a holomorphic $f : U \to X$ has a zero of order $n \in \mathbb{N}$ at $z_0 \in U$ if the Taylor series of f about z_0 has the form

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k$$

for $a_n \neq 0$, i.e., $f^{(j)}(z_0) = 0$ for all $j = 0, 1, \ldots, n-1$ and $f^{(n)}(z_0) \neq 0$.

Proposition 11.6.6. For an open U and $f : U \to X$ holomorphic, if $z_0 \in U$ is a zero of order n for f, then there exists a holomorphic function $g : U \to X$ such that $f(z) = (z - z_0)^n g(z)$ and $g(z_0) \neq 0$, and there exists $\epsilon > 0$ such that $B(z_0, \epsilon) \subset U$ and $f(z) \neq 0$ for all $z \in B(z_0, \epsilon) \setminus \{z_0\}$.

Proof. From the convergent power series we have

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k = (z - z_0)^n \sum_{k=n}^{\infty} a_k (z - z_0)^{k-n} = (z - z_0)^n \sum_{j=0}^{\infty} a_{j+n} (z - z_0)^j,$$

where the last equality is a consequence of the change of index $j = k - n$.

The function

$$g(z) = \sum_{j=0}^{\infty} a_{j+n} (z - z_0)^j$$

is a convergent power series and hence holomorphic.

By hypothesis, $a_n \neq 0$, so that $g(z_0) = a_n \neq 0$.

By the implied continuity of g at z_0, there exists $\epsilon > 0$ such that $B(z_0, \epsilon) \subset U$ and $g(z) \neq 0$ for all $z \in B(z_0, \epsilon)$.

Since the polynomial $(z - z_0)^n$ only equals zero when $z = z_0$, the function $f(z) = (z - z_0)^n g(z)$ does not vanish (meaning does not equal zero) for all $z \in B(z_0, \epsilon) \setminus \{z_0\}$. $\square$

Corollary 11.6.7 (Local Isolation of Zeros). For an open, path-connected U in $\mathbb{C}$, and a holomorphic $f : U \to X$, if there is a sequence $(z_k)_{k=1}^{\infty}$ of distinct points in U where $z_k \to w \in U$ and $f(z_k) = 0$ for all $k \in \mathbb{N}$, then $f(z) = 0$ for all $z \in U$.

Proof. By way of contradiction, suppose that there is a sequence $(z_k)_{k=1}^{\infty}$ of distinct points in U where $z_k \to w \in U$ and $f(z_k) = 0$ for all k, while there exists $z \in U$ such that $f(z) \neq 0$.

Suppose there exists some $\nu > 0$ with $B(w, \nu) \subset U$ such that $f(z) = 0$ for all $z \in B(w, \nu)$.

Then $f^{(j)}(w) = 0$ for all $j = 0, 1, 2, \ldots$, hence by the path-connectedness of U and Corollary 11.6.4 we obtain $f(z) = 0$ for all $z \in U$.

This contradicts the existence of $z \in U$ for which $f(z) \neq 0$.

Hence for all $\nu > 0$ with $B(w, \nu) \subset U$ there exists $z \in B(w, \nu)$ such that $f(z) \neq 0$.

Fix $\nu > 0$ to be the largest value for which $B(w, \nu) \subset U$.

By the convergence of $z_k \to w$ and the continuity of f at w, we have that $f(w) = 0$.

Since there is $z \in B(w, \nu)$ for which $f(z) \neq 0$, the Taylor series for f about w is not identically zero, meaning there exists a smallest $n \in \mathbb{N}$ such that $f^{(n)}(w) \neq 0$.

Thus w is a zero of order n for f.

By Proposition 11.6.6 applied to the zero w of order n, there exists $\epsilon \in (0, \nu)$ for which $f(z) \neq 0$ for all $z \in B(w, \epsilon) \setminus \{w\}$.

By the convergence $z_k \to w$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$ there holds $z_k \in B(w, \epsilon)$.

But since $z_k \in B(w, \epsilon)$ for all $k \geq N$ where $f(z_k) = 0$, this contradicts $f(z) \neq 0$ for all $z \in B(w, \epsilon) \setminus \{w\}$. $\square$

11.6.3 Laurent Series

For a holomorphic $f : U \to X$ such that $f(z_0) \neq 0$ for $z_0 \in U$, the function

$$g(z) = \frac{f(z)}{z - z_0}$$

is not complex differentiable at z_0 and so there is no Taylor series for g about z_0.

We know by Cauchy's Integral formula that for any simple closed contour γ in U enclosing z_0 there holds

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} g(z)\, dz = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0}\, dz.$$

Using the Taylor's series for f about z_0, i.e.,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k,$$

we can express the function g as

$$g(z) = \frac{f(z)}{z-z_0} = \frac{1}{z-z_0}\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-1}$$

which makes sense on the open $U \setminus \{z_0\}$.

Since $z_0 \notin \gamma$ and integration and uniform convergence commute, we can use the series expression for g to compute

$$\oint_\gamma g(z)\ dz = \oint_\gamma \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^{k-1} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}\oint_\gamma (z-z_0)^{k-1}.$$

By Lemma 11.3.5, the contour integral $\oint_\gamma (z-z_0)^{k-1}\ dz = 0$ when $k \geq 1$, and $\oint_\gamma (z-z_0)^{k-1}\ dz = 2\pi i$ when $k = 0$.

Thus we obtain

$$\frac{1}{2\pi i}\oint_\gamma g(z)\ dz = \frac{2\pi i}{2\pi i} f(z_0) = f(z_0)$$

in agreement with Cauchy's Integral Formula.

Definition. For coefficients $a_k \in X$ for $k \in \mathbb{Z}$, a Laurent series is a series of the form

$$\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k.$$

To talk about convergence of Laurent series we will use the open annulus A centered at z_0 with inner radius r and outer radius R defined by

$$A = \{z \in \mathbb{C} : r < |z-z_0| < R\}$$

where $0 \leq r < R \leq \infty$.

Theorem 11.6.8 (Laurent Expansion). For the open annulus A centered at z_0 with inner radius r and outer radius R, if $f : A \to X$ is holomorphic, then f has a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$$

where each of the power series in the decomposition

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k + \sum_{k=1}^{\infty} a_{-k}\left(\frac{1}{z-z_0}\right)^k$$

converge uniformly and absolutely on every compact subannulus

$$D_{\rho,\varrho} = \{z \in \mathbb{C} : \rho \leq |z - z_0| \leq \varrho\}$$

of A, i.e., for all $r < \rho < \varrho < R$. The coefficients a_k in the Laurent series for f are given explicitly by

$$a_k = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z_0)^{k+1}}\ dw$$

for any circle γ of radius strictly between r and R.

See the book for the proof.

Proposition 11.6.9. For an open annulus centered at z_0 with inner radius r and outer radius R, the Laurent series for a holomorphic $f : A \to X$ is unique.

The proof of this is HW (Exercise 11.28). The start of the proof is given in the book.

Remark 11.6.10. Computing the Laurent series is usually quite difficult. As we will see, the only coefficient we really need in the Laurent series of f holomorphic on the annulus $A = B(z_0, \epsilon) \setminus \{0\}$ is that of the term $(z - z_0)^{-1}$ when computing any contour integral $\oint f(z)\ dz$ for a simple closed contour in A that encloses z_0.

Example (in lieu of 11.6.11). The Laurent series of $f(z) = \sin(z)/z^4$ on the open annulus $A = \{z \in \mathbb{C} : 0 < |z| < \infty\} = \mathbb{C} \setminus \{0\}$ is obtained by dividing the power series for $\sin(z)$ by z^4, i.e.,

$$\frac{1}{z^4} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k-3}}{(2k+1)!} = \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{5!} - \cdots .$$

Since the Laurent series for this holomorphic function converges uniformly on A, we can compute the contour integral of f over any circle centered at 0 with radius $\nu > 0$ by a "direct" calculation after commuting the sum and the integral:

$$\oint_\gamma f(z)\ dz = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \oint_\gamma (z - z_0)^{2k-3}\ dz = -\frac{1}{6} \oint_\gamma (z - z_0)^{-1}\ dz = -\frac{\pi i}{3},$$

where all the other contour integrals are zero by Lemma 11.3.5.

This gives

$$\frac{1}{2\pi i} \oint_\gamma f(z)\ dz = -\frac{1}{6}.$$

On the other hand, by Cauchy's Differentiation formula we arrive at the same answer:

$$\frac{1}{2\pi i} \oint_\gamma f(z)\ dz = \frac{1}{3!} \frac{3!}{2\pi i} \oint_\gamma \frac{\sin(z)}{(z-0)^4}\ dz = \frac{1}{6}\big(- \cos(0)\big) = -\frac{1}{6}$$

because the third derivative of $\sin(z)$ is $-\cos(z)$.

Example (in lieu of 11.6.12). Find the Laurent series for

$$f(z) = \frac{2}{(z-1)^2(z+1)}$$

about the point $z_0 = 1$, i.e., an open annulus centered at $z_0 = 1$. [We will determine the inner and outer radius in a moment.]

Applying the method of partial fractions to the function gives

$$f(z) = \frac{1}{(z-1)^2} - \frac{1/2}{z-1} + \frac{1/2}{z+1}.$$

We express the last term as a power series in $(z-1)$ using the geometric series as follows:

$$\begin{aligned}
\frac{1/2}{z+1} &= \frac{1/2}{2-(-z+1)} \\
&= \frac{1/4}{1-(-z+1)/2} \\
&= \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{-z+1}{2}\right)^k \\
&= \frac{1}{4}\sum_{k=0}^{\infty}\left(-\frac{z-1}{2}\right)^k \\
&= \frac{1}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}(z-1)^k \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{k+2}}(z-1)^k.
\end{aligned}$$

We obtain the Laurent series

$$f(z) = \frac{1}{(z-1)^2} - \frac{1/2}{z-1} + \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{k+2}}(z-1)^k$$

on the open annulus A centered at $z_0 = 1$ with inner radius $r = 0$ and outer radius $R = 2$ as determined by the condition for convergence of the geometric series $|(-z+1)/2| < 1$.

For a simple closed contour γ in A that encloses $z_0 = 1$ we use the Laurent series to compute

$$\frac{1}{2\pi i}\oint_{\gamma} f(z)\, dz = -\frac{1}{4\pi i}\oint_{\gamma}\frac{1}{z-1}\, dz = -\frac{2\pi i}{4\pi i} = -\frac{1}{2},$$

where we have used the interchange of integration and uniform convergence, and Lemma 11.3.5.