Math 346 Lecture #30
11.7 The Residue Theorem

The Residue Theorem is the premier computational tool for contour integrals. It includes the Cauchy-Goursat Theorem and Cauchy's Integral Formula as special cases. To state the Residue Theorem we first need to understand isolated singularities of holomorphic functions and quantities called winding numbers.

As always we let $(X, \|\cdot\|_X)$ be a complex Banach space.

11.7.1 Isolated Singularities

Definition 11.7.1. For a point $z_0 \in \mathbb{C}$, an $\epsilon > 0$, and the punctured open disk $U = \{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$, for $f : U \to X$ holomorphic, we say that z_0 is an isolated singularity of f if f is not assumed complex differentiable at z_0.

For an isolated singularity z_0 of f the principal part of the Laurent series

$$\sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$$

of f on $B(z_0, \epsilon) \setminus \{z_0\}$ is the series

$$\sum_{k=-\infty}^{-1} a_k(z - z_0)^k.$$

We use the principal part to classify the isolated singularities.

An isolated singularity z_0 of f is called a removable singularity if the principal part of the Laurent series of f about z_0 is zero, i.e., $a_k = 0$ for all $k = -1, -2, -3, \ldots$.

If f has a removable singularity at z_0, then f extends to a holomorphic function on $B(z_0, \epsilon)$ by means of the power series $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ convergent on $B(z_0, \epsilon)$.

An isolated singular z_0 of f is called a pole of order $N \in \mathbb{N}$ if the principal part of the Laurent series of f about z_0 has the form

$$f(z) = \sum_{k=-N}^{-1} a_k(z - z_0)^k,$$

i.e., $a_k = 0$ for all $k < -N$ in the Laurent series for f about z_0.

A pole of order 1 is called a simple pole.

An isolated singularity z_0 of f is called an essential singularity if the principal part of the Laurent series for f about z_0 has infinitely many nonzero terms, i.e., $a_k \neq 0$ for infinitely many $-k \in \mathbb{N}$.

Example (in lieu of 11.7.2). (i) The function

$$g(z) = \frac{\cos(z) - 1}{z^2} = \frac{1}{z^2}\sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k-2}}{(2k)!} = -\frac{1}{2} + \frac{z^2}{4!} + \cdots$$

has a removable singular at the isolated singularity $z_0 = 0$ of g.

(ii) The function

$$f(z) = \frac{\sin(z)}{z^3} = \frac{1}{z^3}\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!} = \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{5!} + \cdots$$

has a pole of order $N = 3$ at the isolated singularity $z_0 = 0$ of f.

(iii) The function

$$h(z) = \sin\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k (1/z)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k z^{-2k-1}}{(2k+1)!}$$

has an essential singularity at the isolated singularity $z_0 = 0$ of h.

Example 11.7.3. (This is actually a proposition). Suppose f and g are complex-valued holomorphic functions on $B(z_0, \epsilon) \setminus \{z_0\}$ where f has a zero of order k at z_0 and g has a zero of order l at z_0.

By Proposition 11.6.6, there exist complex-valued holomorphic functions F and G on $B(z_0, \epsilon)$ such that $F(z_0) \neq 0$, $G(z_0) \neq 0$, and

$$f(z) = (z - z_0)^k F(z) \text{ and } g(z) = (z - z_0)^l G(z).$$

If $k \geq l$, then the function

$$\frac{f(z)}{g(z)} = \frac{(z - z_0)^k F(z)}{(z - z_0)^l G(z)} = (z - z_0)^{k-l}\frac{F(z)}{G(z)}$$

has a removable singularity at z_0 because $(z - z_0)^{k-l}$ and $F(z)/G(z)$ are holomorphic on $B(z_0, \epsilon)$.

If $k < l$, then f/g has a pole of order $N = l - k$ at z_0 because $G(z) \neq 0$ on a possible smaller ball around z_0.

Definition 11.7.4. For an open set U in $\mathbb{C}$ and finitely many distinct points $z_1, \ldots, z_n$ in U, a function $f : U \setminus \{z_1, \ldots, z_n\} \to X$ is called meromorphic if f is holomorphic on the open set $U \setminus \{z_1, \ldots, z_n\}$ with f having poles at each z_i.

Example 11.7.5. For polynomials p and q with q not identically equal to 0, the rational function $p(z)/q(z)$, in lowest terms (i.e., any common factors that p and q have have already been cancelled), is a meromorphic function on $\mathbb{C} \setminus \{z_1, \ldots, z_k\}$ where $z_1, \ldots, z_k$ are the distinct roots of q.

FYI: It is standard practice is always assume that a rational function is given in lowest terms, unless explicitly told otherwise.

11.7.2 Residues and Winding Numbers

We have already seen that the coefficient a_{-1} of the power $(z - z_0)^{-1}$ in the Laurent series of a function f holomorphic on a punctured disk $B(z_0, \epsilon) \setminus \{z_0\}$ is the quantity needed when computing contour integrals of f on simply closed curves with z_0 in its interior. Because of the important of this coefficient, we give it a name.

Definition 11.7.6. For a holomorphic $f : B(z_0, \epsilon) \setminus \{z_0\} \to X$ and simple close curve γ in $B(z_0, \epsilon) \setminus \{z_0\}$, the quantity

$$\frac{1}{2\pi i} \oint_\gamma f(z)\ dz$$

is called the residue of f at z_0 and is denoted by $\mathrm{Res}(f, z_0)$.

Proposition 11.7.7. If $f : B(z_0, \epsilon) \setminus \{z_0\} \to X$ is holomorphic, then $\mathrm{Res}(f, z_0)$ is the coefficient a_{-1} of the power $(z - z_0)^{-1}$ in the Laurent series of f about z_0.

Proof. Since the Laurent series converges uniformly on compact subsets of $B(z_0, \epsilon) \setminus \{z_0\}$, integration and summation of the Laurent series can be interchanged.

This implies for a simple closed curve γ in $B(z_0, \epsilon) \setminus \{z_0\}$ enclosing z_0 that

$$\begin{aligned}
\mathrm{Res}(f, z_0) &= \frac{1}{2\pi i} \oint_\gamma f(z)\ dz \\
&= \frac{1}{2\pi i} \oint_\gamma \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k\ dz \\
&= \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} a_k \oint_\gamma (z - z_0)^k\ dz \\
&= \frac{a_{-1}}{2\pi i} \oint_\gamma (z - z_0)^{-1}\ dz \\
&= \frac{a_{-1}}{2\pi i} 2\pi i \\
&= a_{-1}
\end{aligned}$$

where we used Lemma 11.3.5. □

The next result provides a characterization of removable singularities, the existence of removable singularity or a pole, and the first glimpse between isolated singularities and residues.

Proposition 11.7.8. Suppose a holomorphic f has an isolated singularity at z_0.

(i) The isolated singularity at z_0 is removable if and only if $\lim_{z \to z_0} f(z)$ exists (as a complex number; the book inaccurately uses the term finite).

(ii) If for some nonnegative integer k the limit $\lim_{z \to z_0} (z - z_0)^k f(z)$ exists (as a complex number), then the isolated singularity z_0 of f is either a removable singularity or a pole of order equal to or less than k.

(iii) If the limit $\lim_{z \to z_0} (z - z_0) f(z)$ exists (as a complex number), then

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

The proof of this is HW (Exercise 11.29).

Up to this point we have mainly use simple closed contour for contour integrals.

The Residue Theorem permits closed contours that can self-intersect and wind around a point many times.

A contour integral on a closed contour depends not only on the residue at a point but also on the number of times the closed contour goes around that point.

Evidence for this is found by the contour integral

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{z-z_0}\,dz$$

for the closed contour $\gamma : [0, 2k\pi] \to \mathbb{C}$ given by $\gamma(t) = z_0 + e^{i\theta}$ for a positive integer k. Computing this contour integral gives

$$\frac{1}{2\pi i}\int_0^{2k\pi} \frac{1}{e^{i\theta}}(ie^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2k\pi} d\theta = \frac{2k\pi}{2\pi} = k.$$

The closed contour γ goes around z_0 in the counterclockwise direction k times while the residue of $1/(z-z_0)$ at z_0 is 1.

If this same curve γ is traversed in the clockwise direction, i.e., $\gamma(\theta) = z_0 + e^{-i\theta}$, then we would get $-k$ as the value of the contour integral.

Furthermore, if γ is closed contour that does not enclose z_0, then $1/(z-z_0)$ is holomorphic on a simply connected open set containing γ but not containing z_0, so that by the Cauchy-Goursat Theorem we have

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{z-z_0}\,dz = 0.$$

These observations motivate the notion of the winding number.

Definition 11.7.9. For a closed contour γ in $\mathbb{C}$ and z_0 a point of $\mathbb{C}$ not on γ, the winding number of γ with respect to z_0 is the quantity

$$I(\gamma, z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-z_0}\,dz.$$

Nota Bene 11.7.10. The winding number essentially counts the total number of times a closed curve traverses counterclockwise around a given point not on the closed curve.

Lemma 11.7.12. For a simply connected open set U in $\mathbb{C}$, a closed contour γ in U, and a point $z_0 \in U$ not on γ, if

$$N(z) = \sum_{k=0}^{\infty} \frac{b_k}{(z-z_0)^k}$$

is uniformly convergent on compact subsets of $U \setminus \{z_0\}$, then there holds

$$\frac{1}{2\pi i}\oint_\gamma N(z)\,dz = \operatorname{Res}(N, z_0) I(\gamma, z_0).$$

The proof of this is HW (Exercise 11.30).

11.7.3 The Residue Theorem

Now that we have the notions of residue and winding number, we can state the Residue Theorem.

Theorem 11.7.13 (The Residue Theorem). For a simply connected U in $\mathbb{C}$ and finitely many points $z_1, \ldots, z_n \in U$, if $f : U \setminus \{z_1, \ldots, z_n\} \to X$ is holomorphic and γ is a closed contour in $U \setminus \{z_1, \ldots, z_n\}$, then

$$\frac{1}{2\pi i}\oint_\gamma f(z)\ dz = \sum_{j=1}^{n} \mathrm{Res}(f, z_j) I(\gamma, z_j).$$

See the book for the proof.

Remark. The Residue Theorem has the Cauchy-Goursat Theorem as a special case. When $f : U \to X$ is holomorphic, i.e., there are no points in U at which f is not complex differentiable, and γ in U is a simple closed curve, we select any $z_0 \in U \setminus \gamma$. The residue of f at z_0 is 0 by Proposition 11.7.8 part (iii), i.e.,

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_1} (z - z_0) f(z) = 0;$$

hence, regardless of the value of $I(\gamma, z_0)$, the Residue Theorem gives

$$\oint_\gamma f(z)\ dz = 0.$$

The Residue Theorem has Cauchy's Integral formula also as special case. When $f : U \to X$ is holomorphic, and $z_0 \in U$, then the function $g(z) = f(z)/(z - z_0)$ is holomorphic on $U \setminus \{z_0\}$, so for any simple closed curve γ in U enclosing z_0 the Residue Theorem gives

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\ dz = \frac{1}{2\pi i}\oint_\gamma g(z)\ dz = \mathrm{Res}(g, z_0) I(\gamma, z_0);$$

here $I(\gamma, z_0) = 1$ because γ is a simple closed curve enclosing z_0, and $\mathrm{Res}(g, z_0) = f(z_0)$ because using the power series for f about z_0 gives the Laurent series

$$g(z) = \frac{f(z)}{z - z_0} = \frac{1}{z - z_0}\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^{k-1}$$

in which the coefficient of $(z - z_0)^{-1}$ is $f(z_0)$.

Using the Residue Theorem requires that we compute the required residues. We have seen two ways to compute the residue of f at a point z_0: by computing the Laurent series of f on $B(z_0, \epsilon) \setminus \{z_0\}$, or by Proposition 11.7.8 part (iii). Of the many other means of computing $\mathrm{Res}(f, z_0)$ we mention a few next.

Proposition 11.7.15. Suppose $g : B(z_0, \epsilon) \to X$ and $h : B(z_0, \epsilon) \to \mathbb{C}$ are holomorphic. If $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$, then the function $g(z)/h(z) : B(z_0, \epsilon) \setminus \{z_0\} \to X$ is meromorphic with a simple pole at z_0 and

$$\mathrm{Res}\left(\frac{g(z)}{h(z)}, z_0\right) = \frac{g(z_0)}{h'(z_0)}.$$

Proof. Since $h(z_0) = 0$, then

$$0 \neq h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{h(z)}{z - z_0}.$$

This implies that

$$\lim_{z \to z_0} \frac{z - z_0}{h(z)} = \frac{1}{h'(z_0)}.$$

With g continuous at z_0 and $g(z_0) \neq 0$ and $h'(z_0) \neq 0$, we obtain

$$0 \neq \frac{g(z_0)}{h'(z_0)} = \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} = \text{Res}\left(\frac{g(z)}{h(z)}, z_0 \right)$$

by Proposition 11.7.8 part (iii).

That g/h has a simple pole at z_0 follows adapting Example 11.7.3.

Since $h(z) = (z - z_0)H(z)$ by Proposition 11.6.6 we have

$$\frac{g(z)}{h(z)} = (z - z_0)^{-1} \frac{g(z)}{H(z)}.$$

The function g/H is holomorphic on some ball $B(z_0, \epsilon)$ and therefore has a power series expansion in powers of $(z - z_0)$.

Multiplying the power series for g/H at z_0 by $(z - z_0)^{-1}$ gives a Laurent series for g/h with nonzero coefficient $\text{Res}(g/h, z_0)$ of $(z - z_0)^{-1}$ and zero coefficients for the powers $(z - z_0)^k$ for all $k = -2, -3, -4, \ldots$. □

Note. While you are responsible for knowing and using Proposition 11.6.6, you are **NOT** responsible for the next two propositions on computing the residue. They are presented so you see a least one method for computing the residue of functions at poles of order 2 and poles of order higher that 2 (and how complicated these residue computations become for poles of order 2 or higher are).

Proposition. Suppose $g : B(z_0, \epsilon) \to X$ and $h : B(z_0, \epsilon) \to 0$ are holomorphic. If $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) = 0$, and $h''(z_0) \neq 0$, then $g(z)/h(z) : B(z_0, \epsilon) \setminus \{z_0\} \to X$ is meromorphic with a pole of order 2 at z_0, and

$$\text{Res}\left(\frac{g(z)}{h(z)}, z_0 \right) = \frac{2g'(z_0)}{h''(z_0)} - \frac{2g(z_0)h^{(3)}(z_0)}{3[h''(z_0)]^2}.$$

Proposition. Suppose $g : B(z_0, \epsilon) \to X$ and $h : B(z_0, \epsilon) \to 0$ are holomorphic. If $g(z_0) \neq 0$, $h^{(j)}(z_0) = 0$ for all $j = 0, 1, \ldots, N-1$, and $h^{(N)}(z_0) \neq 0$, then $g/h : B(z_0, \epsilon) \to X$ has a pole of order N at z_0, and the residue of g/h at z_0 is the product of

$$\left[\frac{N!}{h^{(N)}(z_0)} \right]^N$$

and the symbolic determinant obtained by cofactor expansion along the last column of the $N \times N$ matrix

$$\begin{bmatrix} \frac{h^{(N)}(z_0)}{N!} & 0 & 0 & \cdots & 0 & g(z_0) \\ \frac{h^{(N+1)}(z_0)}{(N+1)!} & \frac{h^{(N)}(z_0)}{N!} & 0 & \cdots & 0 & g^{(1)}(z_0) \\ \frac{h^{(N+2)}(z_0)}{(N+2)!} & \frac{h^{(N+1)}(z_0)}{(N+1)!} & \frac{h^{(N)}(z_0)}{N!} & \cdots & 0 & \frac{g^{(2)}(z_0)}{2!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \frac{h^{(2N-2)}(z_0)}{(2N-2)!} & \frac{h^{(2N-3)}(z_0)}{(2N-3)!} & \frac{h^{(2N-4)}(z_0)}{(2N-4)!} & \cdots & \frac{h^{(N)}(z_0)}{N!} & \frac{g^{(N-2)}(z_0)}{(N-2)!} \\ \frac{h^{(2N-1)}(z_0)}{(2N-1)!} & \frac{h^{(2N-2)}(z_0)}{(2N-2)!} & \frac{h^{(2N-3)}(z_0)}{(2N-3)!} & \cdots & \frac{h^{(N+1)}(z_0)}{(N+1)!} & \frac{g^{(N-1)}(z_0)}{(N-1)!} \end{bmatrix}.$$

Remark. Unfortunately for an essential singularity of f at z_0 there are no "simple" formulas for computing the residue of f at z_0. We typically rely on finding the Laurent series for f at z_0 to find its residue at z_0.

Example (in lieu of 11.7.16). For the holomorphic function $f(z) = 1/(z^2 + 1)$ the numerator is $g(z) = 1$ and the denominator is $h(z) = z^2 + 1$.

The roots of $h(z) = (z - i)(z + i)$ are $z_1 = i$ and $z_2 = -i$, i.e., $h(z_1) = 0$ and $h(z_2) = 0$.

Since $h'(z) = 2z$ we have $h'(z_1) = 2i \neq 0$ and $h'(z_2) = -2i \neq 0$.

By Proposition 11.7.15, the function f has a simple pole at each of z_1 and z_2 where

$$\mathrm{Res}(f, z_1) = \frac{g(z_1)}{h'(z_1)} = \frac{1}{2i} \text{ and } \mathrm{Res}(f, z_2) = \frac{g(z_2)}{h'(z_2)} = -\frac{1}{2i}.$$

The simple closed contour $\gamma = \{z \in \mathbb{C} : |z| = 2\}$, i.e., the circle centered at 0 with radius 2, encloses both simple poles of f.

For the winding numbers we have $I(\gamma, z_1) = 1$ and $I(\gamma, z_2) = 1$.

By the Residue Theorem we compute

$$\frac{1}{2\pi i} \oint_\gamma f(z)\, dz = \sum_{j=1}^{2} \mathrm{Res}(f, z_j) I(\gamma, z_j) = \frac{1}{2i} - \frac{1}{2i} = 0.$$

Example 11.7.17. We now show how to use the Residue Theorem to compute the value of improper real-valued integrals of the form

$$\int_{-\infty}^{\infty} f(x)\, dx.$$

We will do this for

$$f(x) = \frac{1}{1 + x^4}.$$

The improper integral of f over $\mathbb{R}$ converges by a comparison test with $1/(1+x^2)$, i.e., since $1+x^4 \geq 1+x^2$, then

$$0 \leq \frac{1}{1+x^4} \leq \frac{1}{1+x^2}$$

and the improper integral of $1/(1+x^2)$ converges because

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\ dx = \lim_{R\to\infty} \arctan(x)\Big|_{-R}^{R} = \pi < \infty.$$

Convergence of the improper integral of $1/(1+x^4)$ over $\mathbb{R}$ justifies writing

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\ dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\ dx.$$

We recognize that the integrand is equal to the complex-valued function

$$f(z) = \frac{1}{1+z^4}$$

when $z \in \mathbb{R}$.

The function $f(z)$ is complex differentiable except at the four roots of the denominator $h(z) = 1+z^4$.

We can find these four roots using Euler's Formula as follows.

By writing $-1 = e^{i\pi+2in\pi}$ for an arbitrary integer n, the equation $1+z^4 = 0$ becomes $e^{i\pi+2in\pi} = z^4$.

Taking fourth roots of both sides of this equation gives $e^{i\pi/4+ni\pi/2} = z$.

The root complex roots of $h(z) = z^4+1$ are correspond to the four distinct angles $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$ in $[0, 2\pi)$; the four roots are

$$z_1 = e^{i\pi/4}, z_2 = e^{3i\pi/4}, z_3 = e^{5i\pi/4}, z_4 = e^{7i\pi/4}.$$

There is one root in each quadrant of the complex plane.

The function f is meromorphic on $\mathbb{C} \setminus \{z_1, z_2, z_3, z_4\}$.

Since $h'(z) = 4z^3$ and $h'(z_j) \neq 0$ for all $j = 1, 2, 3, 4$, each point z_j is a simple pole for $f(z) = 1/h(z)$ with residue

$$\text{Res}(f, z_j) = \frac{1}{h'(z_j)}.$$

Now for the "magic" of the Residue Theorem.

For $R \geq 2$, form the closed simple contour D that is the sum of the line γ from $-R$ to R and the top half C of the circle with center 0 and radius R traversed counterclockwise.

This gives

$$\oint_D f(z) dz = \int_\gamma f(z)\ dz + \int_C f(z)\ dz = \int_{-R}^{R} \frac{1}{1+x^4}\ dx + \int_C f(z)\ dz.$$

The contour D encloses two simple poles of $f(z)$, the two in the first and second quadrant. The residues of f at these poles are

$$\mathrm{Res}(f, z_1) = \frac{1}{4(e^{i\pi/4})^3} = \frac{1}{4e^{3i\pi/4}} \text{ and } \mathrm{Res}(f, z_2) = \frac{1}{4(e^{3i\pi/4})^3} = \frac{1}{4e^{9i\pi/4}}.$$

The winding numbers of D at the poles are $I(D, z_j) = 1$ for $j = 1, 2$.

By the Residue Theorem we have

$$\begin{aligned}
\oint_D \frac{1}{1+z^4}\, dz &= 2\pi i \left[\mathrm{Res}\left(\frac{1}{1+z^4}, z_1\right) + \mathrm{Res}\left(\frac{1}{1+z^4}, z_2\right)\right] \\
&= 2\pi i \left[\frac{1}{4e^{3i\pi/4}} + \frac{1}{4e^{9i\pi/4}}\right] \\
&= \frac{\pi i}{2}\left[e^{-3i\pi/4} + e^{-9i\pi/4}\right] \\
&= \frac{\pi i}{2}\left[e^{-3i\pi/4} + e^{-i\pi/4}\right] \\
&= \frac{\pi i}{2}\left[\cos(3\pi/4) - i\sin(3\pi/4) + \cos(\pi/4) - i\sin(\pi/4)\right] \\
&= \frac{\pi i}{2}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] \\
&= \frac{\pi i}{2}\left(-\frac{2i}{\sqrt{2}}\right) \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

By the parameterization $\xi(\theta) = Re^{i\theta}$, $\theta \in [0, \pi]$, of C we obtain

$$\begin{aligned}
\left|\int_C \frac{1}{1+z^4}\, dz\right| &= \left|\int_0^\pi \frac{iRe^{i\theta}}{1+R^4e^{4i\theta}}\, d\theta\right| \\
&\le \int_0^\pi \left|\frac{iRe^{i\theta}}{1+R^4e^{4i\theta}}\right| d\theta \\
&= \int_0^\pi \frac{R}{|1+R^4e^{4i\theta}|}\, d\theta \\
&\le \int_0^\pi \frac{R}{|R^4e^{4i\theta}| - 1}\, d\theta \\
&= \frac{R\pi}{R^4 - 1},
\end{aligned}$$

where for the last inequality we have used the "reverse" triangle inequality

$$|R^4e^{4i\theta}| - |-1| \le |R^4e^{4i\theta} - (-1)|.$$

The upper bound on the norm of the contour integral of $f(z)$ over C goes to 0 as $R \to \infty$, and this implies that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\, dx = \oint_D f(z)\, dz = \frac{\pi}{\sqrt{2}}.$$