Math 346 Lecture #31
12.1 Projections

Before, in Chapter 3, we used an inner product on a vector space V to define an orthogonal projection P of V onto a subspace X of V. An algebraic property of an orthogonal projection P is that $P^2 = P$ (called idempotence). This property is actually sufficient to define a projection on a vector space not necessarily equipped with an inner product.

Throughout this note, we assume that V is a vector space over $\mathbb{F}$.

12.1.1 Projections

Definition 12.1.1. A linear operator $P \in \mathscr{L}(V)$ is called a projection if $P^2 = P$.

Example 12.1.2. If $P \in \mathscr{L}(V)$ is a projection, then so is $I - P \in \mathscr{L}(V)$, where $I \in \mathscr{L}(V)$ is the identity operator defined by $I(v) = v$ for all $v \in V$.

You have it as HW (Exercise 12.1) to show that $I - P$ is a projection.

The linear operator $I - P$ is called the complementary projection of P.

Lemma 12.1.3. If $P \in \mathscr{L}(V)$ is a projection, then

(i) $\mathrm{y} \in \mathscr{R}(P)$ if and only if $P\mathrm{y} = \mathrm{y}$, and
(ii) $\mathscr{N}(P) = \mathscr{R}(I - P)$.

Proof. (i) If $P\mathrm{y} = \mathrm{y}$, then $y \in \mathscr{R}(P)$.

On the other hand, if $y \in \mathscr{R}(P)$, then there exists $\mathrm{x} \in V$ such that $\mathrm{y} = P\mathrm{x}$.

Since $P^2 = P$ we have $P\mathrm{y} = P^2\mathrm{x} = P\mathrm{x} = \mathrm{y}$.

(ii) We have $\mathrm{x} \in \mathscr{N}(P)$ if and only if $P\mathrm{x} = 0$.

We also have $P\mathrm{x} = 0$ if and only if $(I - P)\mathrm{x} = \mathrm{x} - P\mathrm{x} = \mathrm{x}$.

Because $I - P$ is a projection by Example 12.1.2, we have by part (i) that $(I - P)\mathrm{x} = \mathrm{x}$ if and only if $\mathrm{x} \in \mathscr{R}(I - P)$.

Thus we have that $\mathrm{x} \in \mathscr{N}(P)$ if and only if $\mathrm{x} \in \mathscr{R}(I - P)$. □

Remark. Because $I - P$ is a projection when P is a projection, we can apply part (ii) of Lemma 12.1.3 to $I - P$ to get $\mathscr{N}(I - P) = \mathscr{R}(P)$.

We show next that a projection P on V decomposes V into the direct sum of two complementary subspaces of V.

Theorem 12.1.4. If $P \in \mathscr{L}(V)$ is a projection, then $V = \mathscr{R}(P) \oplus \mathscr{N}(P)$.

Proof. For every $\mathrm{x} \in V$ there holds

$$P\mathrm{x} + (I - P)\mathrm{x} = P\mathrm{x} + I\mathrm{x} - P\mathrm{x} = \mathrm{x}.$$

By Lemma 12.1.3, part (ii), we have $\mathscr{R}(I - P) = \mathscr{N}(P)$, so that $(I - P)\mathrm{x} \in \mathscr{N}(P)$.

Thus each $\mathrm{x} \in V$ is the sum of $P\mathrm{x} \in \mathscr{R}(P)$ and $(I - P)\mathrm{x} \in \mathscr{N}(P)$.

This implies that $V = \mathscr{R}(P) + \mathscr{N}(P)$.

To show that $\mathscr{R}(P) \cap \mathscr{N}(P) = \{0\}$ we take $x \in \mathscr{R}(P) \cap \mathscr{N}(P)$.

Then $\mathrm{x} \in \mathscr{R}(P)$, so by Lemma 12.1.3 part (i) we have $P\mathrm{x} = \mathrm{x}$.

But as $\mathrm{x} \in \mathscr{N}(P)$, we have $P\mathrm{x} = 0$.

Thus $0 = P\mathrm{x} = \mathrm{x}$, giving $V = \mathscr{R}(P) \oplus \mathscr{N}(P)$. □

Corollary 12.1.5. For $\dim(V) < \infty$, if $P \in \mathscr{L}(V)$ is a projection with $S = [\mathrm{s}_1, \ldots, \mathrm{s}_k]$ a basis for $\mathscr{R}(P)$ and $T = [\mathrm{t}_1, \ldots, \mathrm{t}_l]$ a basis for $\mathscr{N}(P)$, then $S \cup T$ is a basis for V (i.e., $k + l = \dim(V)$) and the block matrix representation of P in the basis $S \cup T$ is

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where I is the $k \times k$ identity matrix, and each 0 is a zero matrix of appropriate size.

Proof. By Theorem 12.1.4 the union $S \cup T$ is a basis for V.

By Theorem 4.2.9, the block matrix representation of P in the basis $S \cup T$ is

$$\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

where A_11 is the $k \times k$ matrix representation of P on $\mathscr{R}(P)$, and A_{22} is the $l \times l$ matrix representation of P on $\mathscr{N}(P)$.

By Lemma 12.1.3 part (i), P is the identity map on $\mathscr{R}(P)$ so that $A_{11} = I$.

Since $P\mathrm{x} = 0$ for each $\mathrm{x} \in \mathscr{N}(P)$, we have $A_{22} = 0$. □

Theorem 12.1.6. For subspaces W_1 and W_2 of V (not assumed finite dimensional), if $V = W_1 \oplus W_2$, then there exists a unique projection $P \in \mathscr{L}(V)$ such that $\mathscr{R}(P) = W_1$ and $\mathscr{N}(P) = W_2$.

Proof. From $V = W_1 \oplus W_2$, there is for each $\mathrm{x} \in V$ unique $\mathrm{x}_1 \in W_1$ and unique $\mathrm{x}_2 \in W_2$ such that $\mathrm{x} = \mathrm{x}_1 + \mathrm{x}_2$.

Define $P : V \to V$ by $P\mathrm{x} = \mathrm{x}_1$.

Since $\mathrm{x}_1 = \mathrm{x}_1 + 0$ for $\mathrm{x}_1 \in W_1$, then $P\mathrm{x}_1 = \mathrm{x}_1$.

Hence $P^2\mathrm{x}_1 = P(P\mathrm{x}_1) = P\mathrm{x}_1 = \mathrm{x}_1$, so that $P^2 = P$ on W_1.

Since $\mathrm{x}_2 = 0 + \mathrm{x}_2$ for $\mathrm{x}_2 \in W_2$, then $P\mathrm{x}_2 = 0$.

Hence $P^2\mathrm{x}_2 = P(P\mathrm{x}_2) = P0 = 0$ (since $0 \in W_1$).

Thus $P^2 = P$ on $V = W_1 \oplus W_2$.

To show that P is linear, we let $\mathrm{x} = \mathrm{x}_1 + \mathrm{x}_2$ and $\mathrm{y} = \mathrm{y}_1 + \mathrm{y}_2$, where $\mathrm{x}_1, \mathrm{y}_1 \in W_1$ and $\mathrm{x}_2, \mathrm{y}_2 \in W_2$.

Then for $a, b \in \mathbb{C}$ we have

$$a\mathrm{x} + b\mathrm{y} = (a\mathrm{x}_1 + b\mathrm{y}_1) + (a\mathrm{x}_2 + b\mathrm{y}_2)$$

where $a\mathrm{x}_1 + b\mathrm{y}_1 \in W_1$ and $a\mathrm{x}_2 + b\mathrm{y} \in W_2$ because W_1 and W_2 are subspaces.

This gives

$$\begin{aligned} P(a\mathrm{x} + b\mathrm{y}) &= P\big((a\mathrm{x}_1 + b\mathrm{y}) + (a\mathrm{x}_2 + b\mathrm{y}_2)\big) \\ &= a\mathrm{x}_1 + b\mathrm{y}_1 \\ &= aP(\mathrm{x}) + bP(\mathrm{y}). \end{aligned}$$

Thus P is linear with $\mathscr{R}(P) = W_1$ and $\mathscr{N}(P) = W_2$.

If $Q \in \mathscr{L}(V)$ is another projection with $\mathscr{R}(Q) = W_1$ and $\mathscr{N}(Q) = W_2$, then for all $\mathrm{x} = \mathrm{x}_1 + \mathrm{x}_2 \in W_1 \oplus W_2$ there holds

$$\mathrm{Px} = P\mathrm{x}_1 + P\mathrm{x}_2 = \mathrm{x}_1 = Q\mathrm{x}_1 = Q\mathrm{x}_1 + Q\mathrm{x}_2 = Q\mathrm{x}.$$

This shows that $P = Q$ on V, so that P is unique. □

Definition. The unique projection $P \in \mathscr{L}(V)$ associated to $V = W_1 \oplus W_2$ in Theorem 12.1.6 is called the projection onto W_1 along W_2.

For a projection $P \in \mathscr{L}(V)$, we have by Theorem 12.1.4 that $V = \mathscr{R}(P) \oplus \mathscr{N}(P)$, so that with $W_1 = \mathscr{R}(P)$ and $W_2 = \mathscr{N}(P)$, the projection P is the unique projection onto $\mathscr{R}(P)$ along $\mathscr{N}(P)$.

We sometimes says that a projection P is a projection onto $\mathscr{R}(P)$ without reference to along $\mathscr{N}(P)$ because the along part is always given by $\mathscr{N}(P)$.

Keep in mind that there do exist distinct projections $P, Q \in \mathscr{L}(V)$ with $\mathscr{R}(P) = \mathscr{R}(Q)$ but $\mathscr{N}(P) \neq \mathscr{N}(Q)$. For example, the projections $P, Q \in \mathscr{L}(\mathbb{C}^2)$ defined by $P(e_1) = e_1$, $P(e_2) = 0$, $Q(e_1) = e_1$, and $Q(e_1 + e_2) = 0$ has the same range but different kernels.

Remark. In a finite dimensional inner product space V, the projection P onto W_1 along W_2 is an orthogonal projection only when $W_2 = W_1^\perp$. In an infinite dimensional inner product space, a projection P onto W_1 along W_2 is an orthogonal projection only when W_1 is a closed subspace and $W_2 = W_1^\perp$.

Example (in lieu of 12.1.7). Consider the vector space $V = C([0, 1], \mathbb{C})$ equipped with the inner product

$$\langle f, g\rangle = \int_0^1 \overline{f(t)}g(t)\ dt.$$

Define the operator $P : V \to V$ by $P(f)$ is the constant function from $[0, 1]$ to $\mathbb{C}$ with value $f(0)$.

The operator P is linear because for $f, g \in V$ and $a, b \in \mathbb{C}$ there holds

$$P(af + bg) = af(0) + bg(0) = aP(f) + bP(g).$$

The operator $P \in \mathscr{L}(V)$ is a projection because for all $f \in V$ there holds

$$P^2(f) = P(f(0)) = f(0) = P(f).$$

The subspace $\mathscr{R}(P)$ consists of the constant functions from $[0, 1]$ to $\mathbb{C}$.

The subspace $\mathscr{N}(P)$ consists of those continuous functions $f : [0,1] \to \mathbb{C}$ such that $f(0) = 0$.

By Theorem 12.1.4 there holds $V = \mathscr{R}(P) \oplus \mathscr{N}(P)$, i.e., each function $f \in V$ can be written uniquely as

$$f(t) = f(0) + (f(t) - f(0))$$

for $f(0) \in \mathscr{R}(P)$ and $f(t) - f(0) \in \mathscr{N}(P)$.

With $W_1 = \mathscr{R}(P)$ and $W_2 = \mathscr{N}(P)$, we have by Theorem 12.1.6 that P is the unique projection onto W_1 along W_2.

Is P an orthogonal projection? That is, is W_1 closed and is $W_1^{\perp} = W_2$?

The answer is no for the second condition because there exists $f \in W_1$ and $g \in W_2$ such that $\langle f, g\rangle \neq 0$, i.e., for $f = 1$ and $g(t) = t$ we have

$$\langle f, g\rangle = \int_0^1 t\, dt = 1/2 \neq 0.$$

Note. Sometimes nonorthogonal projections, such as in the previous example, are called oblique projections.

12.1.2 Invariant Subspaces and Their Projections

Recall from Section 4.2 that a subspace W of V is invariant for $L \in \mathscr{L}(V)$ or that W is L-invariant if $L(W) \subset W$.

Theorem 12.1.8. For $L \in \mathscr{L}(V)$, a subspace W of V is L-invariant if and only if for any projection $P \in \mathscr{L}(V)$ onto W there holds

$$LP = PLP.$$

Proof. Suppose W is L-invariant.

Let $P \in \mathscr{L}(L)$ be a projection with $\mathscr{R}(P) = W$.

By the Remark after Lemma 12.1.3 the projection $I-P$ satisfies $\mathscr{N}(I-P) = \mathscr{R}(P) = W$.

For each $\mathrm{w} \in W$ the L-invariance of W implies that $L\mathrm{w} \in W$, and hence for all $\mathrm{w} \in W$ that $(I - P)Lw = 0$.

Since $P\mathrm{v} \in W$ for all $\mathrm{v} \in V$, we obtain $(I - P)LP\mathrm{v} = 0$ for all $\mathrm{v} \in V$.

This implies that $(I - P)LP = 0$ or rewritten that $LP = PLP$.

Now suppose for a projection P onto W that $LP = PLP$.

Since $P\mathrm{w} = \mathrm{w}$ for all $\mathrm{w} \in W$ and $LP = PLP$, it follows that

$$L\mathrm{w} = LP\mathrm{w} = PLP\mathrm{w} = PL\mathrm{w}.$$

Since $PL\mathrm{w} \in W$, we obtain $L\mathrm{w} = PL\mathrm{w} \in W$, whence W is L-invariant. □

Theorem 12.1.9. Suppose W_1, W_2 are subspaces of V for which $V = W_1 \oplus W_2$, and $L \in \mathscr{L}(V)$. Then W_1 and W_2 are both L-invariant if and only if the projection P onto W_1 along W_2 satisfies $LP = PL$.

Proof. Suppose both W_1 and W_2 are L-invariant.

Since $V = W_1 \oplus W_2$, each $\mathrm{v} \in V$ can be written uniquely as $\mathrm{v} = \mathrm{w}_1 + \mathrm{w}_2$ for $\mathrm{w}_1 \in W_1$ and $\mathrm{w}_2 \in W_2$.

We have $P\mathrm{w}_1 = \mathrm{w}_1$ and $P\mathrm{w}_2 = 0$, and so $LP\mathrm{w}_2 = 0$.

By the L-invariance of W_1 and W_2 we have $L\mathrm{w}_1 \in W_1$ and $L\mathrm{w}_2 \in W_2$.

Since P is the projection onto W_1 along W_2 there holds $PL\mathrm{w}_1 = L\mathrm{w}_1$ and $PL\mathrm{w}_2 = 0$.

Thus

$$PL\mathrm{v} = PL\mathrm{w}_1 + PL\mathrm{w}_2 = PL\mathrm{w}_1 = L\mathrm{w}_1 = LP\mathrm{w}_1 = LP\mathrm{w}_1 + LP\mathrm{w}_2 = LP\mathrm{v}.$$

This holds for all $\mathrm{v} \in V$ so that $LP = PL$.

Now suppose that $LP = PL$ for the projection P onto W_1 along W_2.

Then $\mathscr{R}(P) = W_1$ and $\mathscr{N}(P) = W_2$.

For $w_1 \in W_1$ we have

$$L\mathrm{w}_1 = LP\mathrm{w}_1 = PL\mathrm{w}_1 \in W_1.$$

This shows that W_1 is L-invariant.

Because $LP = PL$ there holds

$$L(I - P) = L - LP = L - PL = (I - P)L.$$

By Lemma 12.1.3 part (ii), we have $\mathscr{R}(I - P) = \mathscr{N}(P) = W_2$.

For $\mathrm{w}_2 \in W_2$ we have $P\mathrm{w}_2 = 0$ so that

$$L\mathrm{w}_2 = L(I - P)\mathrm{w}_2 = (I - P)L\mathrm{w}_2 \in W_2.$$

This shows that W_2 is L-invariant. □

12.1.3 Eigenprojections for Simple Operators

We apply the theory of projections to a simple operator on a finite dimensional vector space, where the range of the projections are the eigenspaces of that simple operator. This will give a decomposition of a simple operator on a finite dimensional vector space into a sum of scalar multiples of the projections, where the scalars are the corresponding eigenvalues.

Recall for $i, j = 1, \ldots, n$ that δ_{ij} is the $(i, j)^{\text{th}}$ entry of the $n \times n$ identity matrix I.

Proposition 12.1.10. Suppose $A \in M_n(\mathbb{C})$ is a simple operator whose distinct (complex) eigenvalues are $\lambda_1, \ldots, \lambda_n$. Let $S \in M_n(\mathbb{C})$ be the matrix whose columns are the corresponding right eigenvectors of A, and denote the i^{th} column of S by r_i. Let $\ell_1^{\mathrm{T}}, \ldots, \ell_n^{\mathrm{T}}$ be the corresponding left eigenvectors of A, i.e., the rows of S^{-1}. Define the $n \times n$ matrices $P_k = \mathrm{r}_k \ell_k^{\mathrm{T}}$, $k = 1, \ldots, n$. Then

(i) $\ell_i^{\mathrm{T}} \mathrm{r}_j = \delta_{ij}$ for all $i, j = 1, \ldots, n$,

(ii) $P_i P_j = \delta_{ij} P_i$ for all $i, j = 1, \ldots, n$,

(iii) $P_iA = AP_i = \lambda_i P_i$ for all $i = 1, \ldots, n$,

(iv) $\sum_{i=1}^n P_i = I$, and

(v) $A = \sum_{i=1}^n \lambda_i P_i$.

Proof. (i) Since $\mathrm{r}_1, \ldots, \mathrm{r}_n$ are columns of S, since $\ell_1^{\mathrm{T}}, \ldots, \ell_n^{\mathrm{T}}$ are the rows of S^{-1}, and since $S^{-1}S = I$ we have $\ell_i^{\mathrm{T}}\mathrm{r}_j = \delta_{ij}$.

(ii) Computing we have

$$P_iP_j = \mathrm{r}_i\ell_i^{\mathrm{T}}\mathrm{r}_j\ell_j^{\mathrm{T}} = \mathrm{r}_i\delta_{ij}\ell_j^{\mathrm{T}} = \delta_{ij}\mathrm{r}_i\ell_j^{\mathrm{T}} = \begin{cases} \mathrm{r}_i\ell_i^{\mathrm{T}} & \text{if } j = i, \\ 0 & \text{if } i \neq j. \end{cases}$$

Since $\mathrm{r}_i\ell_i^{\mathrm{T}} = P_i$, we obtain $P_iP_j = \delta_{ij}P_i$.

(iii) Computing we have

$$P_iA = \mathrm{r}_i\ell_i^{\mathrm{T}}A = \mathrm{r}_i\lambda_i\ell_i^{\mathrm{T}} = \lambda_i\mathrm{r}_i\ell_i^{\mathrm{T}} = \lambda_iP_i$$

and

$$AP_i = A\mathrm{r}_i\ell_i^{\mathrm{T}} = \lambda_i\mathrm{r}_i\ell_i^{\mathrm{T}} = \lambda_iP_i$$

thus giving $P_iA = AP_i = \lambda_iP_i$.

(iv) We notice that

$$\sum_{i=1}^n P_i = \sum_{i=1}^n \mathrm{r}_i\ell_i^{\mathrm{T}}$$

is the outer product expansion of SS^{-1} obtained by partitioning S into columns and S^{-1} into rows.

Since $SS^{-1} = I$, we obtain $\sum_{i=1}^\infty P_i = I$.

(v) Using (iii) and (iv) we compute

$$\sum_{i=1}^n \lambda_iP_i = \sum_{i=1}^n AP_i = A\sum_{i=1}^n P_i = AI = A$$

giving the result. □

Remark. The matrices P_i are projections by part (ii) of Proposition 12.1.10 because $P_i^2 = P_iP_i = \delta_{ii}P_i = P_i$. The rank of each of these projections is one because the columns of P_i are all scalar multiples of the nonzero right eigenvector r_i. Indeed the range of P is the one-dimensional eigenspace of A corresponding to the eigenvalue λ_i.

Definition. For a simple operator $A \in M_n(\mathbb{C})$ the rank-1 projections $P_1, \ldots, P_n$ in Proposition 12.1.10 are called the eigenprojections of A.

Example (in lieu of 12.1.11). The eigenvalues and right eigenvectors of the simple

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \in M_2(\mathbb{C})$$

are

$$\lambda_1 = 3,\ \mathrm{r}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \lambda_2 = -1,\ \mathrm{r}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The matrix of right eigenvectors

$$S = \begin{bmatrix} \mathrm{r}_1 & \mathrm{r}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

has inverse

$$S^{-1} = -\frac{1}{4}\begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}.$$

The rows of S^{-1} give left eigenvectors of A:

$$\ell_1^{\mathrm{T}} = \frac{1}{4}\begin{bmatrix} 2 & 1 \end{bmatrix},\ \ell_2^{\mathrm{T}} = \frac{1}{4}\begin{bmatrix} 2 & -1 \end{bmatrix}.$$

The eigenprojections are

$$P_1 = \mathrm{r}_1\ell_1^{\mathrm{T}} = \frac{1}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

and

$$P_2 = \mathrm{r}_2\ell_2^{\mathrm{T}} = \frac{1}{4}\begin{bmatrix} 1 \\ -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}.$$

Each of P_1 and P_2 has rank 1, and we can verify the properties listed in Proposition 12.1.10.

For property (ii) we have

$$\begin{aligned} P_1P_2 &= \frac{1}{16}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0, \\ P_1^2 &= \frac{1}{16}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 8 & 4 \\ 16 & 8 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = P_1, \\ P_2^2 &= \frac{1}{16}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 8 & -4 \\ -16 & 8 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} = P_2. \end{aligned}$$

For property (iii) we have

$$\begin{aligned} AP_1 &= \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6 & 3 \\ 12 & 6 \end{bmatrix}, \\ P_1A &= \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6 & 3 \\ 12 & 6 \end{bmatrix} = \frac{3}{4}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \lambda_1P_1 \\ AP_2 &= \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}, \\ P_2A &= \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} = \lambda_2P_2. \end{aligned}$$

For property (iv) we have

$$P_1 + P_2 = \frac{1}{4}\left\{\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\right\} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I.$$

Finally for property (v) we have

$$\lambda_1 P_1 + \lambda_2 P_2 = \frac{3}{4}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = A.$$