# Math 346 Lecture #34
## 12.4 Spectral Resolution

We begin by using the resolvent $R_A(z) = (zI - A)^{-1}$ of a linear operator $A \in M_n(\mathbb{C})$ to construct spectral projections for $A$.

**Definition 12.4.1.** For $A \in M_n(\mathbb{C})$, let $\lambda \in \sigma(A)$ and $\Gamma$ be a positively oriented simple closed curve enclosing $\lambda$ but no other elements of $\sigma(A)$. The spectral projection or eigenprojection of $A$ associated with $\lambda$ is defined to be

$$P_\lambda = \operatorname{Res}(R_A(z), \lambda) = \frac{1}{2\pi i} \oint_\Gamma R_A(z)\ dz.$$

**Theorem 12.4.2.** For $A \in M_n(\mathbb{C})$, the spectral projections $P_\lambda$, $\lambda \in \sigma(A)$, have the following properties.

(i) Idempotency: $P_\lambda^2 = P_\lambda$ for all $\lambda \in \sigma(A)$.

(ii) Independence: $P_\lambda P_{\lambda'} = 0 = P_{\lambda'} P_\lambda$ for all $\lambda, \lambda' \in \sigma(A)$ with $\lambda \neq \lambda'$.

(iii) $A$-invariance: $AP_\lambda = P_\lambda A$ for all $\lambda \in \sigma(A)$.

(iv) Completeness: $\displaystyle\sum_{\lambda \in \sigma(A)} P_\lambda = I$.

Proof. (i) Let $\Gamma$ and $\Gamma'$ be two positively oriented simple closed curves enclosing $\lambda$ and no other elements of $\sigma(A)$.

WLOG we can assume that $\Gamma$ is in the interior of $\Gamma'$.

By the definition of the winding number, we have for each $z \in \Gamma$ that

$$\oint_{\Gamma'} \frac{dz'}{z' - z} = 2\pi i \text{ and } \oint_\Gamma \frac{dz}{z' - z} = 0.$$

By path-independence of the contour integral we get two different expressions for the same spectral projection:

$$\frac{1}{2\pi i} \oint_\Gamma R_A(z)\ dz = P_\lambda = \frac{1}{2\pi i} \oint_{\Gamma'} R_A(z')\ dz'.$$

Hence we can write

$$\begin{aligned} P_\lambda^2 &= \left( \frac{1}{2\pi i} \oint_\Gamma R_A(z)\ dz \right) \left( \frac{1}{2\pi i} \oint_{\Gamma'} R_A(z')\ dz' \right) \\ &= \left( \frac{1}{2\pi i} \right)^2 \oint_\Gamma \left( \oint_{\Gamma'} R_A(z')\ dz' \right) R_A(z)\ dz \\ &= \left( \frac{1}{2\pi i} \right)^2 \oint_\Gamma \oint_{\Gamma'} R_A(z) R_A(z')\ dz' dz, \end{aligned}$$

where we have used $R_A(z)R_A(z') = R_A(z')R_A(z)$.

Using Hilbert's Identity $R_A(z) - R_A(z') = (z' - z)R_A(z)R_A(z')$ we have

$$\begin{aligned}
P_\lambda^2 &= \left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \oint_{\Gamma'} \frac{R_A(z) - R_A(z')}{z' - z}\, dz' dz \\
&= \left(\frac{1}{2\pi i}\right)^2 \left[\oint_\Gamma R_A(z)\left(\oint_{\Gamma'} \frac{dz'}{z'-z}\right) dz - \oint_{\Gamma'} R_A(z')\left(\oint_\Gamma \frac{dz}{z'-z}\right) dz'\right] \\
&= \frac{1}{2\pi i}\oint_\Gamma R_A(z)\, dz \\
&= P_\lambda,
\end{aligned}$$

where we switched the order of integration (i.e., Fubini's Theorem) in the second equality.

This shows that each $P_\lambda$ is an idempotent.

(ii) Let $\Gamma$ and $\Gamma'$ be two positively oriented simple closed curves in $\rho(A)$ enclosing $\lambda$ and $\lambda'$ respectively for which no other points of $\sigma(A)$ are enclosed by either curve.

Then for $z \in \Gamma$ and $z' \in \Gamma'$ we have

$$\oint_{\Gamma'} \frac{dz'}{z'-z} = 0 \text{ and } \oint_\Gamma \frac{dz}{z'-z} = 0.$$

The composition of the two spectral projections

$$P_\lambda = \frac{1}{2\pi i}\oint_\Gamma R_A(z)\, dz \text{ and } P_{\lambda'} = \frac{1}{2\pi i}\oint_{\Gamma'} R_A(z')\, dz'$$

is

$$P_\lambda P_{\lambda'} = \left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \oint_{\Gamma'} R_A(z)R_A(z')\, dz' dz.$$

By Hilbert's Identity and Fubini's Theorem we have

$$\begin{aligned}
P_\lambda P_{\lambda'} &= \left(\frac{1}{2\pi i}\right)^2 \oint_\Gamma \oint_{\Gamma'} \frac{R_A(z) - R_A(z')}{z' - z}\, dz' dz \\
&= \left(\frac{1}{2\pi i}\right)^2 \left[\oint_\Gamma R_A(z)\left(\oint_{\Gamma'} \frac{dz'}{z'-z}\right) dz - \oint_{\Gamma'} R_A(z')\left(\oint_\Gamma \frac{dz}{z'-z}\right) dz'\right] \\
&= 0.
\end{aligned}$$

This gives the independence of the spectral projections.

(iii) By Lemma 12.3.5 part (iii), we have $AR_A(z) = R_A(z)A$ for all $z \in \rho(A)$.

This implies that

$$A\left[\oint_\Gamma R_A(z)\, dz\right] = \oint_\Gamma AR_A(z)\, dz = \oint_\Gamma R_A(z)A\, dz = \left[\oint_\Gamma R_A(z)\, dz\right] A.$$

Thus $AP_\lambda = P_\lambda A$.

(iv) Let $\Gamma$ be a positively oriented circle centered at $z = 0$ and having radius $R > r(A)$.

Theorems 12.3.8 and 12.3.14 guarantee that the Laurent series

$$R_A(z) = \sum_{k=0}^{\infty} \frac{A^k}{z^{k+1}}$$

holds along $\Gamma$.

By the interchanging of integration and uniform convergence of the Laurent series on compact sets such as $\Gamma$ we have

$$\frac{1}{2\pi i} \oint_{\Gamma} R_A(z)\ dz = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \oint_{\Gamma} \frac{A^k}{z^{k+1}}\ dz = A^0 = I.$$

On the other hand, for positively oriented simple closed curves $\Gamma_\lambda$, $\lambda \in \sigma(A)$, each $\Gamma_\lambda$ enclosing $\lambda$ and no other elements of $\sigma(A)$, we have by the Cauchy-Goursat Theorem and the appropriate cuts that

$$\frac{1}{2\pi i} \oint_{\Gamma} R_A(z)\ dz = \frac{1}{2\pi i} \sum_{\lambda \in \sigma(A)} \oint_{\Gamma_\lambda} R_A(z)\ dz = \sum_{\lambda \in \sigma(A)} P_\lambda.$$

This gives the completeness of the spectral projections. □

Remark 12.4.3. The double integrals in the previous proof required Fubini's Theorem to switch the order of integration. Although we stated Fubini's Theorem for real-valued functions, it readily extends to Banach-space valued functions.

Example (in lieu of 12.4.4). Find the spectral projections for

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

Recall that the resolvent for this matrix is

$$R_A(z) = \frac{1}{(z-3)(z+1)} \begin{bmatrix} z-1 & 1 \\ 4 & z-1 \end{bmatrix}.$$

From the partial fraction decompositions for the entries of $R_A(z)$,

$$\frac{z-1}{(z-3)(z+1)} = \frac{1/2}{z-3} + \frac{1/2}{z+1},$$
$$\frac{1}{(z-3)(z+1)} = \frac{1/4}{z-3} + \frac{-1/4}{z+1},$$
$$\frac{4}{(z-3)(z+1)} = \frac{1}{z-3} + \frac{-1}{z+1},$$

we obtain the partial fraction decomposition

$$R_A(z) = \frac{1}{z-3} \begin{bmatrix} 1/2 & 1/4 \\ 1 & 1/2 \end{bmatrix} + \frac{1}{z+1} \begin{bmatrix} 1/2 & -1/4 \\ -1 & 1/2 \end{bmatrix}.$$

Recall that we can express $1/(z-3)$ as a geometric series in $(z+1)$, and $1/(z+1)$ as a geometric series in $(z-3)$, giving the Laurent series of $R_A(z)$ about $z=3$ and about $z=-1$.

From the partial fraction decomposition of $R_A(z)$ we have the spectral projections,

$$P_3 = \text{Res}(R_A(z), 3) = \begin{bmatrix} 1/2 & 1/4 \\ 1 & 1/2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix},$$

and

$$P_{-1} = \text{Res}(R_A(z), -1) = \begin{bmatrix} 1/2 & -1/4 \\ -1 & 1/2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}.$$

These spectral projection $P_3$ and $P_{-1}$ are precisely the projections $P_i = \mathrm{r}_i \ell_i^{\mathrm{T}}$ we computed in Lecture #31 by way of right eigenvectors

$$\mathrm{r}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \mathrm{r}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

of $A$, and the left eigenvectors

$$\ell_1^{\mathrm{T}} = \frac{1}{4}\begin{bmatrix} 2 & 1 \end{bmatrix}, \ \ell_2^{\mathrm{T}} = \frac{1}{4} = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

of $A$ that satisfy $\ell_i^{\mathrm{T}} \mathrm{r}_j = \delta_{ij}$.

We already have seen that these projections are idempotent, independent, commute with $A$, and sum to $I$.

Example. Find the spectral projections for

$$A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix}.$$

Recall from Lecture #31 that the resolvent for this matrix is

$$R(z) = \begin{bmatrix} (z-6)^{-1} & (z-6)^{-2} & 7(z-6)^{-2}(z-4)^{-1} \\ 0 & (z-6)^{-1} & 7(z-6)^{-1}(z-4)^{-1} \\ 0 & 0 & (z-4)^{-1} \end{bmatrix}.$$

From the partial fraction decompositions

$$\frac{7}{(z-6)^2(z-4)} = \frac{-7/4}{z-6} + \frac{7/2}{(z-6)^2} + \frac{7/4}{(z-4)},$$
$$\frac{7}{(z-6)(z-4)} = \frac{7/2}{z-6} + \frac{-7/2}{z-4},$$

we obtain the partial fraction decomposition

$$R_A(z) = \frac{1}{z-6}\begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{(z-6)^2}\begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{z-4}\begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

From this we obtain

$$P_6 = \operatorname{Res}(R_A(z), 6) = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$P_4 = \operatorname{Res}(R_A(z), 4) = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

We check the four properties of Theorem 12.4.2.

The matrices $P_6$ and $P_4$ are idempotent:

$$P_6^2 = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = P_6,$$

and

$$P_4^2 = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} = P_4.$$

The matrices $P_6$ and $P_4$ are independent:

$$P_6 P_4 = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} = 0,$$

and

$$P_4 P_6 = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

The matrices $P_6$ and $P_4$ commute with $A$:

$$P_6 A = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -7 \\ 0 & 6 & 21 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A P_6 = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -7 \\ 0 & 6 & 21 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_4 A = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 7 \\ 0 & 0 & -14 \\ 0 & 0 & 4 \end{bmatrix},$$

and

$$A P_4 = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 7 \\ 0 & 0 & -14 \\ 0 & 0 & 4 \end{bmatrix}.$$

The matrices $P_6$ and $P_4$ are complete: they sum to $I$.

Does $6P_6 + 4P_4 = A$ equal? No because

$$6P_6 + 4P_4 = \begin{bmatrix} 6 & 0 & -7/2 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix} \neq A = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 7 \\ 0 & 0 & 4 \end{bmatrix}.$$

This should not be surprising since $A$ is not diagonalizable.

But notice that

$$A - 6P_6 - 4P_4 = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Have we seen this matrix before? Yes, we saw it in the resolvent,

$$R_A(z) = \frac{1}{z-6}\begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{(z-6)^2}\begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{z-4}\begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix},$$

the coefficient matrix of the $(z-6)^{-2}$ term. Is this just a coincidence? We shall see.

**Example (in lieu of 12.4.5).** Find the spectral projections for

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Recall that we computed the resolvent of this matrix in Lecture #33, where we obtained

$$\det(zI - A) = (z-2)^3(z-5).$$

and

$$\operatorname{adj}(zI - A) = \begin{bmatrix} (z-2)^2(z-5) & (z-2)(z-5) & z-5 & 3 \\ 0 & (z-2)^2(z-5) & (z-2)(z-5) & 3(z-2) \\ 0 & 0 & (z-2)^2(z-5) & 3(z-2)^2 \\ 0 & 0 & 0 & (z-2)^3 \end{bmatrix}.$$

The resolvent is thus

$$R_A(z) = \begin{bmatrix} (z-2)^{-1} & (z-2)^{-2} & (z-2)^{-3} & 3(z-2)^{-3}(z-5)^{-1} \\ 0 & (z-2)^{-1} & (z-2)^{-2} & 3(z-2)^{-2}(z-5)^{-1} \\ 0 & 0 & (z-2)^{-1} & 3(z-2)^{-1}(z-5)^{-1} \\ 0 & 0 & 0 & (z-5)^{-1} \end{bmatrix}.$$

From the partial fraction decompositions

$$\frac{3}{(z-2)^3(z-5)} = \frac{-1/9}{z-2} + \frac{-1/3}{(z-2)^2} + \frac{-1}{(z-2)^3} + \frac{1/9}{z-5},$$

$$\frac{3}{(z-2)^2(z-5)} = \frac{-1/3}{z-2} + \frac{-1}{(z-2)^2} + \frac{1/3}{z-5},$$

$$\frac{3}{(z-2)(z-5)} = \frac{-1}{z-2} + \frac{1}{z-5},$$

we compute

$$P_2 = \text{Res}(R_A(z), 2) = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$P_5 = \text{Res}(R_A(z), 5) = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We check the four properties of Theorem 12.4.2 for the matrices $P_2$ and $P_5$.

The matrices $P_2$ and $P_5$ are idempotent:

$$P_2^2 = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = P_2,$$

and

$$P_5^2 = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = P_5.$$

The matrices $P_2$ and $P_5$ are independent:

$$P_2 P_5 = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 0$$

and

$$P_5 P_2 = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0.$$

The matrices $P_2$ and $P_5$ commute with $A$:

$$AP_2 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & -5/9 \\ 0 & 2 & 1 & -5/3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_2 A = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & -5/9 \\ 0 & 2 & 1 & -5/3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$AP_5 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 5/9 \\ 0 & 0 & 0 & 5/3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

and

$$P_5A = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 5/9 \\ 0 & 0 & 0 & 5/3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

The matrices $P_2$ and $P_5$ are complete: they sum to $I$.

Since $A$ is not diagonalizable, the sum $2P_2 + 5P_5$ does not equal $A$. Something else is needed that comes from the resolvent $R_A(z)$, that will be fully explored in the next two sections.

Before that we learn how to represent the holomorphic image of a matrix in terms of a contour integral and the resolvent, and some of its consequences.

Theorem 12.4.6 (Spectral Resolution Theorem). For $A \in M_n(\mathbb{C})$, if the power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

has a radius of convergence $b > r(A)$, then for any positively oriented simple closed contour $\Gamma$ containing $\sigma(A)$ and contained within the disk $B(0, b_0)$ for some $b_0 \in (b, r(A))$, there holds

$$f(A) = \frac{1}{2\pi i} \oint_\Gamma f(z) R_A(z)\ dz.$$

Proof. WLOG we can assume that $\Gamma$ is the circle centered at 0 with radius $b_0$.

Using the Laurent series for $R_A(z)$ on $|z| > r(A)$, we have

$$\frac{1}{2\pi i} \oint_\Gamma f(z) R_A(z)\ dz = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z} \sum_{k=0}^{\infty} \frac{A^k}{z^k}\ dz.$$

Since $f(z)/z$ is bounded on $\Gamma$ and the summation converges uniformly on compact sets, the sum and integral can be interchanged to give

$$\frac{1}{2\pi i} \oint_\Gamma f(z) R_A(z)\ dz = \sum_{k=0}^{\infty} A^k \left[ \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z^{k+1}}\ dz \right].$$

By Cauchy's Differentiation formula we have

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z^{k+1}}\ dz.$$

Thus we obtain

$$\frac{1}{2\pi i} \oint_\Gamma f(z) R_A(z)\ dz = \sum_{k=0}^{\infty} a_k A^k = f(A).$$

This give the result. $\square$

Corollary 12.4.7. The spectral radius of $A \in M_n(\mathbb{C})$ satisfies

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

Proof. We saw in Lecture #33 that $r(A) \geq \sigma_M = \sup\{|\lambda| : \lambda \in \sigma(A)\}$ as a consequence of the Laurent series of $R_A(z)$ (see Theorems 12.3.8 and 12.3.14).

To get $r(A) = \sigma_M$ it suffices to show that $r(A) \leq \sigma_M + \epsilon$ for all $\epsilon > 0$.

To this end, let $\Gamma$ be a positive oriented circle centered at 0 with radius $\sigma_M + \epsilon$.

Thus for $z \in \Gamma$ we have $|z| = \sigma_M + \epsilon$.

The contour $\Gamma$ is compact and the resolvent $R_A(z)$ is continuous on $\Gamma$ so that for any matrix norm $\|\cdot\|$ on $M_n(\mathbb{C})$ we have

$$K = \sup\{\|R_A(z)\| : z \in \Gamma\} < \infty\}.$$

By the Spectral Resolution formula applied to the entire $f(z) = z^n$, we have

$$A^n = \frac{1}{2\pi i} \oint_\Gamma z^n R_A(z)\ dz.$$

Applying the matrix norm to this gives

$$\|A^n\| \leq \frac{1}{2\pi}(\sigma_M + \epsilon)^n K \oint_\Gamma |dz| \leq K(\sigma_M + \epsilon)^{n+1}$$

since the arclength of $\Gamma$ is $2\pi(\sigma_M + \epsilon)$.

This gives

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} \leq \lim_{n\to\infty} K^{1/n}(\sigma_M + \epsilon)^{1+1/n} = \sigma_M + \epsilon.$$

This gives the result. □

Corollary 12.4.8 (Cayley-Hamilton Theorem). For $A \in M_n(\mathbb{C})$, let $p(z) = \det(zI - A)$, the characteristic polynomial of $A$. Then $p(A) = 0$.

Proof. Let $\Gamma$ be a simple closed contour containing $\sigma(A)$.

By the Spectral Resolution formula applied to the entire function $p(z)$ we have

$$p(A) = \frac{1}{2\pi i} \oint_\Gamma p(z) R_A(z)\ dz.$$

By Cramer's Rule,

$$R_A(z) = \frac{1}{\det(zI - A)} \mathrm{adj}(zI - A) = \frac{1}{p(z)} \mathrm{adj}(zI - A).$$

Thus

$$p(A) = \frac{1}{2\pi i} \oint_\Gamma p(z) \frac{1}{p(z)} \mathrm{adj}(zI - A)\ dz = \frac{1}{2\pi i} \oint_\Gamma \mathrm{adj}(zI - A)\ dz.$$

Each entry of the adjugate of $zI - A$ is a polynomial in $z$, and hence the $\mathrm{adj}(zI - A)$ is entire.

By the Cauchy-Goursat Theorem we obtain $p(A) = 0$. □.

**Example (in lieu of 12.4.10).** We verify the Cayley-Hamilton Theorem for

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

The characteristic polynomial of $A$ is

$$p(z) = (z-3)(z+1) = z^2 - 2z - 3.$$

Checking we have

$$\begin{aligned} A^2 - 2A - 3I &= \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 2 \\ 8 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 8 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$