Math 346 Lecture #35
12.5 Spectral Decomposition I

The goal of this and the next section is to establish the existence of a spectral decomposition of any linear operator on $\mathbb{C}^n$. Such a spectral decomposition depends on the spectral projections which are the residues of the resolvent at the eigenvalues. But such a spectral decomposition also depends, as we saw in the last lecture, on other terms in the Laurent series of the resolvent about the eigenvalues. We begin to develop a better understanding of the Laurent series of the resolvent in this section.

For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, there exist $A_k \in M_n(\mathbb{C})$, $k \in \mathbb{Z}$, (depending on λ) such that the resolvent of A as a Laurent series about λ has the form

$$R_A(z) = \sum_{k=-\infty}^{\infty} A_k (z-\lambda)^k.$$

By the Laurent Expansion Theorem, we have for each $k \in \mathbb{Z}$ that

$$A_k = \frac{1}{2\pi i} \oint_\Gamma \frac{R_A(z)}{(z-\lambda)^{k+1}}\, dz$$

for a positively oriented simple closed contour Γ enclosing λ but no other element of $\sigma(A)$. The coefficient A_{-1} is the spectral projection P_λ. We are going to discover the nature of the relationships that exist among all the coefficient matrices A_k in Laurent series for $R_A(z)$ about λ.

Nota Bene 12.5.1. Be aware that A_k is a coefficient matrix in a Laurent series while A^k is the k^{th} power of A.

Notation. For $n \in \mathbb{Z}$, define

$$\eta_n = \begin{cases} 1 & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

This is a characteristic or indicator function on the set $\mathbb{Z}$.

Lemma 12.5.2. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, let Γ and Γ' be two positively oriented simple closed contours in $\rho(A)$ enclosing λ and no other element of $\sigma(A)$. Assume further that Γ is in the interior of Γ', that $z' \in \Gamma'$, and $z \in \Gamma$. Then for all $m \in \mathbb{N}$ there holds

$$\text{(i)}\ \frac{1}{2\pi i} \oint_\Gamma (z-\lambda)^{-m-1}(z'-z)^{-1}\, dz = \eta_m (z'-\lambda)^{-m-1},$$

and for all $n \in \mathbb{N}$ there holds

$$\text{(ii)}\ \frac{1}{2\pi i} \oint_{\Gamma'} (z'-\lambda)^{-n-1}(z'-z)^{-1}\, dz' = (1-\eta_n)(z-\lambda)^{-n-1}.$$

Proof. (i) Since every point z' on Γ' is outside of Γ, the function $(z'-z)^{-1}$ is holomorphic within Γ.

Using the geometric series, we expand $(z'-z)^{-1}$ in terms of $z-\lambda$:

$$\frac{1}{z'-z} = \frac{1}{z'-\lambda}\cdot\frac{1}{1-\left(\frac{z-\lambda}{z'-\lambda}\right)} = \frac{1}{z'-\lambda}\sum_{k=0}^{\infty}\left(\frac{z-\lambda}{z'-\lambda}\right)^k = \sum_{k=0}^{\infty}\frac{(z-\lambda)^k}{(z'-\lambda)^{k+1}}.$$

WLOG we may shrink Γ to a small circle Γ_λ centered at λ with every point on Γ_λ closer to λ than z'.

For a fixed $m \in \mathbb{N}$ we then have

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-\lambda)^{-m-1}(z'-z)^{-1}\,dz\\
&\quad= \frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-\lambda)^{-m-1}\left[\sum_{k=0}^{\infty}\frac{(z-\lambda)^k}{(z'-\lambda)^{k+1}}\right]\,dz\\
&\quad= \sum_{k=0}^{\infty}(z'-\lambda)^{-k-1}\left[\frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-\lambda)^{-m-1+k}\,dz\right].
\end{aligned}$$

By Lemma 11.3.5 we have

$$\frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-\lambda)^{-m-1+k}\,dz = \begin{cases}1 & \text{if } k=m,\\ 0 & \text{if } k\neq m.\end{cases}$$

When $m<0$, the case $k=m$ never occurs since $k\geq 0$, thus giving

$$\frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-\lambda)^{-m-1}(z'-z)^{-1}\,dz = 0.$$

When $m\geq 0$, only one of the integrals is nonzero, and that occurs when $k=m$, thus giving

$$\frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-\lambda)^{-m-1}(z'-z)^{-1}\,dz = (z'-\lambda)^{-m-1}.$$

These two outcomes for the integral combine through the function η_m to give

$$\frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-\lambda)^{-m-1}(z'-z)^{-1}\,dz = \eta_m(z'-\lambda)^{-m-1}.$$

(ii) In this case, both λ and z lies inside Γ'.

Using the Cauchy-Goursat Theorem, the same circle Γ_λ from part (i), a small circle Γ_z centered at z such that Γ_z does not intersect Γ', and the appropriate cuts, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_{\Gamma'}(z'-\lambda)^{-n-1}(z'-z)^{-1}\,dz'\\
&\quad= \frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z'-\lambda)^{-n-1}(z'-z)^{-1}\,dz'\\
&\qquad+ \frac{1}{2\pi i}\oint_{\Gamma_z}(z'-\lambda)^{-n-1}(z'-z)^{-1}\,dz'.
\end{aligned}$$

For the former integral we have

$$\frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z'-\lambda)^{-n-1}(z'-z)^{-1}\,dz'$$
$$=-\frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z'-\lambda)^{-n-1}(z-z')^{-1}\,dz'=-\eta_n(z-\lambda)^{-n-1}$$

by an argument similar to that in part (i) with Γ replaced with Γ' and z and z' switched. For the latter integral we have by the Cauchy Integral formula that

$$\frac{1}{2\pi i}\oint_{\Gamma_z}\frac{(z'-\lambda)^{-n-1}}{z'-z}\,dz'=(z-\lambda)^{-n-1}$$

since $z'\to(z'-\lambda)^{-n-1}$ is holomorphic on a simply connected open set containing Γ_z. Combining the two integrals gives the result. □

Lemma 12.5.3. The matrix coefficients A_k in the Laurent expansion

$$R_A(z)=\sum_{k=-\infty}^{\infty}A_k(z-\lambda)^k$$

about $\lambda\in\sigma(A)$ satisfy

$$A_mA_n=(1-\eta_m-\eta_n)A_{m+n+1}.$$

Proof. Let Γ and Γ' be two positively oriented simply closed contours enclosing $\lambda\in\sigma(A)$ but no other element of $\sigma(A)$, and further assume that Γ is in the interior of Γ'.

For $m,n\in\mathbb{N}$ we write

$$A_m=\frac{1}{2\pi i}\oint_{\Gamma}\frac{R_A(z)}{(z-\lambda)^{m+1}}\,dz,\;\; A_n=\frac{1}{2\pi i}\oint_{\Gamma'}\frac{R_A(z')}{(z'-\lambda)^{n+1}}\,dz'.$$

Using properties of the resolvent in Lemma 12.3.5, and Fubini's Theorem, we have

$$\begin{aligned}A_mA_n&=\left(\frac{1}{2\pi i}\right)^2\oint_{\Gamma}\oint_{\Gamma'}(z-\lambda)^{-m-1}(z'-\lambda)^{-n-1}R_A(z')R_A(z)\,dz'dz\\&=\left(\frac{1}{2\pi i}\right)^2\oint_{\Gamma}\oint_{\Gamma'}(z-\lambda)^{-m-1}(z'-\lambda)^{-n-1}\frac{R_A(z)-R_A(z')}{z'-z}\,dz'dz\\&=\left(\frac{1}{2\pi i}\right)^2\oint_{\Gamma}(z-\lambda)^{-m-1}R(z)\left[\oint_{\Gamma'}(z'-\lambda)^{-n-1}(z'-z)^{-1}\,dz'\right]\,dz\\&\quad-\left(\frac{1}{2\pi i}\right)^2\oint_{\Gamma'}(z'-\lambda)^{-n-1}R(z')\left[\oint_{\Gamma}(z-\lambda)^{-m-1}(z'-z)^{-1}\,dz\right]\,dz'\\&=\frac{1}{2\pi i}\oint_{\Gamma}(z-\lambda)^{-m-1}R(z)(1-\eta_n)(z-\lambda)^{-n-1}\,dz\\&\quad-\frac{1}{2\pi i}\oint_{\Gamma'}(z'-\lambda)^{-n-1}R(z')\eta_m(z'-\lambda)^{-m-1}\,dz'\end{aligned}$$

The integrand of the second integral has an isolated singularity at λ, and so by the Cauchy-Goursat Theorem and the appropriate cut we can replace Γ' with Γ and z' with z without changing the the value of the second integral.

This gives for A_nA_m the expression

$$\begin{aligned}
&\frac{1}{2\pi i}\oint_\Gamma \left[(z-\lambda)^{-m-1}R(z)(1-\eta_n)(z-\lambda)^{-n-1} - (z-\lambda)^{-n-1}R(z)\eta_m(z-\lambda)^{-m-1}\right] dz \\
&= \frac{1}{2\pi i}\oint_\Gamma (z-\lambda)^{-m-1}(z-\lambda)^{-n-1}R(z)\big[1-\eta_n-\eta_m\big]\ dz \\
&= [1-\eta_m-\eta_n]\frac{1}{2\pi i}\oint_\Gamma (z-\lambda)^{-m-n-2}R(z)\ dz \\
&= [1-\eta_m-\eta_n]A_{m+n+1}.
\end{aligned}$$

This gives the identity $A_mA_n = (1-\eta_m-\eta_n)A_{m+n+1}$. □

Remark 12.5.4. Since $P_\lambda = A_{-1}$, Lemma 12.5.3 gives another proof that

$$P_\lambda^2 = A_{-1}A_{-1} = (1-\eta_{-1}-\eta_{-1})A_{-1-1+1} = A_{-1} = P_\lambda.$$

Notation. To express the relationships that exists among the coefficient matrices A_k in the Laurent series of $R_A(z)$ about λ, we define

$$D_\lambda = A_{-2} \text{ and } S_\lambda = A_0.$$

Lemma 12.5.5. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, there holds

(i) $A_{-n} = D_\lambda^{n-1}$ for all $n \geq 2$,
(ii) $A_n = (-1)^n S_\lambda^{n+1}$ for all $n \geq 1$,
(iii) the spectral projection P_λ commutes with D_λ and with S_λ, where in particular,

$$P_\lambda D_\lambda = D_\lambda,\ \ P_\lambda S_\lambda = 0,$$

(iv) The Laurent series of $R_A(z)$ about λ is

$$R_A(z) = \frac{P_\lambda}{z-\lambda} + \sum_{k=1}^\infty \frac{D_\lambda^k}{(z-\lambda)^{k+1}} + \sum_{k=0}^\infty (-1)^k(z-\lambda)^k S_\lambda^{k+1},$$

(v) the spectral projection P_λ commutes with $R_A(z)$, where in particular

$$P_\lambda R_A(z) = \frac{P_\lambda}{z-\lambda} + \sum_{k=1}^\infty \frac{D_\lambda^k}{(z-\lambda)^{k+1}}.$$

The proof of these is HW (Exercises 12.23, 12.24, and 12.25).

Remark. The Laurent series for $R_A(z)$ about $\lambda \in \sigma(A)$ is completely determined by three matrices $P_\lambda = A_{-1}$, $D_\lambda = A_{-2}$, and $S_\lambda = A_0$.

Example. We verify some parts of Lemma 12.5.5 for the linear operator

$$A = \begin{bmatrix} 6 & 1 & 0 \\ 6 & 1 & 7 \\ 0 & 0 & 4 \end{bmatrix}$$

and use other parts of Lemma 12.5.5 to compute Laurent series expansion of $R_A(z)$ about $\lambda = 6$.

We computed previously that

$$R_A(z) = \frac{1}{z-6}\begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{(z-6)^2}\begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{z-4}\begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix},$$

so that the spectral projections are

$$P_6 = \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } P_4 = \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Also from the partial fraction decomposition of $R_A(z)$ we have

$$D_6 = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } D_4 = 0.$$

We may thus neatly write

$$R_A(z) = \frac{P_6}{z-6} + \frac{D_6}{(z-6)^2} + \frac{P_4}{z-4}.$$

Verifying part (iii) of Lemma 12.5.5, the matrices P_6 and D_6 satsify

$$\begin{aligned} P_6 D_6 &= \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = D_6 \\ &= \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 7/2 \\ 0 & 0 & 0 \end{bmatrix} = D_6 P_6. \end{aligned}$$

The matrix D_6 satisfies

$$D_6^2 = \begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Thus $D_6^k = 0$ for all $k \geq 2$, so that

$$\sum_{k=1}^{\infty} \frac{D_6^k}{(z-6)^{k+1}} = \frac{D_6}{(z-6)^2}.$$

We could compute S_6 by writing $1/(z-4)$ as a geometric series in $(z-6)$.

Instead we make use of parts (iv) and (v) of Lemma 12.5.5. First, by part (iv) we have

$$\sum_{k=0}^{\infty}(-1)^k(z-6)^k S_6^{k+1} = R_A(z) - \left(\frac{P_6}{z-6} + \frac{D_6}{(z-6)^2}\right).$$

By part (v) we have

$$R_A(z)P_6 = \frac{P_6}{z-6} + \frac{D_6}{(z-6)^2}.$$

Combining these gives

$$\sum_{k=0}^{\infty}(-1)^k(z-6)^k S_6^{k+1} = R_A(z) - R_A(z)P_6 = R_A(z)(I-P_6) = R_A(z)P_4$$

by the completeness $P_6 + P_4 = I$.

In the product $R_A(z)P_4$ we have $P_6P_4 = 0$ and $P_4^2 = 0$, but what is D_6P_4? It is

$$\begin{bmatrix} 0 & 1 & 7/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 7/4 \\ 0 & 0 & -7/2 \\ 0 & 0 & 1 \end{bmatrix} = 0.$$

Is this just a coincidence? According to part (v) of Lemma 12.5.5 it is not!

We thus have

$$R_A(z)P_4 = \frac{P_4}{z-4}.$$

The point of all of this is that we have

$$\sum_{k=0}^{\infty}(-1)^k(z-6)^k S_6^{k+1} = \frac{P_4}{z-4}.$$

Evaluating this equality at $z = 6$ gives

$$S_6 = \frac{P_4}{2}.$$

Since $P_6P_4 = 0$, we verify part (iii) of Lemma 12.5.5 in that $P_6S_6 = S_6P_6 = 0$.

Since $P_4^2 = P_4$, we obtain $S_6^{k+1} = (1/2)^{k+1}P_4$, thus giving the Laurent series of the resolvent about $\lambda = 6$, namely

$$R_A(z) = \frac{D_6}{(z-6)^2} + \frac{P_6}{z-6} + P_4 \sum_{k=0}^{\infty} \frac{(-1)^k(z-6)^k}{2^{k+1}}.$$

Using the geometric series one can verify that

$$\sum_{k=0}^{\infty} \frac{(-1)^k (z-6)^k}{2^{k+1}} = \frac{1}{z-4}.$$

Example (in lieu of 12.5.6). We compute the Laurent series

$$R_A(z) = \sum_{k=-\infty}^{\infty} A_k (z-2)^k$$

for the linear operator

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

To this end we need P_2, D_2, and S_2.

We computed previously that

$$R_A(z) = \frac{1}{z-2} \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{(z-2)^2} \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \frac{1}{(z-2)^3} \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{z-5} \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The spectral projections are

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } P_5 = \begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We also have

$$D_2 = A_{-2} = \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and by way of verification that

$$D_2^2 = \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_{-3}.$$

To find S_2 we have by part (iv) of Lemma 12.5.5 that

$$\sum_{k=0}^{\infty}(-1)^k(z-2)^k S_2^{k+1} = R_A(z) - \left(\frac{P_2}{z-2} + \frac{D_2}{(z-2)^2}\right)$$

and by part (v) of Lemma 12.5.5 that

$$P_2 R_A(z) = \frac{P_2}{z-2} + \frac{D_2}{(z-2)^2}.$$

Combining these gives

$$\sum_{k=0}^{\infty}(-1)^k(z-2)^k S_2^{k+1} = R_A(z) - P_2 R_A(z) = (I - P_2)R_A(z) = R_A(z)P_5 = \frac{P_5}{z-5},$$

where we have used the completeness $P_2 + P_5 = I$ and part (iv) of Lemma 12.5.5 applied to $\lambda = 5$.

Evaluation of the equality at $z = 2$ gives $S_2 = -(1/3)P_5$.

[Note the book incorrectly says to integrate to get this for the example it considers.]

Thus the Laurent series for the resolvent of A around $\lambda = 2$ is

$$R_A(z) = \frac{D_2^2}{(z-2)^3} + \frac{D_2}{(z-2)^2} + \frac{P_2}{z-2} - P_5\sum_{k=0}^{\infty}\frac{(z-2)^k}{3^{k+1}}.$$

Using the geometric series we can verify that

$$-\sum_{k=0}^{\infty}\frac{(z-2)^k}{3^{k+1}} = \frac{1}{z-5}.$$

We mentioned previously that $A \neq 2P_2 + 5P_5$ since A is not semisimple, but that something else was happening.

The spectral decomposition of A is $2P_2 + D_2 + 5P_2$ because

$$\begin{aligned} 2P_2 + D_2 + 5P_5 &= 2\begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 5\begin{bmatrix} 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = A. \end{aligned}$$