Math 346 Lecture #36
12.6 Spectral Decomposition II

We are now in the position of proving the existence of the spectral decomposition

$$A = \sum_{\lambda \in \sigma(A)} \big(\lambda P_\lambda + D_\lambda\big)$$

for every linear operator $A \in M_n(\mathbb{C})$. This spectral decomposition includes the one for semisimple linear operators. We have already seen two examples of the spectral decomposition for non-semisimple linear operators. We will also prove that the range of each spectral projection P_λ is precisely the generalized eigenspace, and formalize the simplified partial fraction form of the resolvent we have seen many times. We begin with understanding the linear operator D_λ that appears in the spectral decomposition.

Lemma 12.6.1. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, the linear operator D_λ (the matrix coefficient A_{-2} in the Laurent series of $R_A(z)$ about λ) satisfies

$$D_\lambda = (A - \lambda I)P_\lambda.$$

Moreover, the spectral radius of D_λ is zero.

Proof. The equation $D_\lambda = (A - \lambda I)P_\lambda$ holds if and only if

$$AP_\lambda = \lambda P_\lambda + D_\lambda$$

holds, so it suffices to verify that latter equation.

Let Γ_λ be a positively oriented circle centered at λ with small enough radius to exclude other elements of $\sigma(A)$.

By definition of the resolvent, we have $R_A(z)(zI - A) = I$, from which we get

$$zR_A(z) = AR(z) + I$$

where we have used $AR_A(z) = R_A(z)A$ from Lemma 12.3.5.

Since

$$P_\lambda = \frac{1}{2\pi i}\oint_{\Gamma_\lambda} R_A(z)\,dz \text{ and } \frac{1}{2\pi i}\oint_{\Gamma_\lambda} I(z)\,dz = 0,$$

where $I(z) = z$ is entire, we have that

$$\begin{aligned}
AP_\lambda &= \frac{1}{2\pi i}\oint_{\Gamma_\lambda} AR_A(z)\,dz \\
&= \frac{1}{2\pi i}\oint_{\Gamma_\lambda} AR_A(z)\,dz + \frac{1}{2\pi i}\oint_{\Gamma_\lambda} I\,dz \\
&= \frac{1}{2\pi i}\oint_{\Gamma_\lambda} \big(AR_A(z) + I(z)\big)\,dz \\
&= \frac{1}{2\pi i}\oint_{\Gamma_\lambda} zR_A(z)\,dz.
\end{aligned}$$

By writing $z = (z-\lambda)+\lambda$ we obtain

$$\begin{aligned}
AP_\lambda &= \frac{1}{2\pi i}\oint_{\Gamma_\lambda} \lambda R_A(z)\ dz + \frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-\lambda)R_A(z)\ dz \\
&= \frac{\lambda}{2\pi i}\oint_{\Gamma_\lambda} R_A(z)\ dz + \frac{1}{2\pi i}\oint_{\Gamma_\lambda}\frac{R_A(z)}{(z-\lambda)^{-2+1}}\ dz \\
&= \lambda A_{-1} + A_{-2} \\
&= \lambda P_\lambda + D_\lambda.
\end{aligned}$$

To show that $r(D_\lambda) = 0$, we parameterize Γ_λ by $z(t) = \lambda + \rho e^{it}$ for $0 < \rho < 1$ with ρ small enough so that Γ_λ encloses λ but no other element of $\sigma(A)$.

By Lemma 12.5.5 part (i) we have that $A_{-(k+1)} = D_\lambda^k$ for all $n \geq 1$.

Now

$$A_{-(k+1)} = \frac{1}{2\pi i}\oint_{\Gamma_\lambda}\frac{R_A(z)}{(z-\lambda)^{-k}}\ dz.$$

Thus for any matrix norm $\|\cdot\|$ on $M_n(\mathbb{C})$ we have

$$\begin{aligned}
\|D_\lambda^k\| &= \left\|\frac{1}{2\pi i}\oint_{\Gamma_\lambda}(z-\lambda)^k R_A(z)\ dz\right\| \\
&\frac{1}{2\pi}\left\|\int_0^{2\pi}\rho^k e^{ikt}R_A(\lambda+\rho e^{it})\rho i e^{it}\ dt\right\| \\
&\leq \rho^{k+1}\sup\{\|R(z)\| : z\in\Gamma_\lambda\}.
\end{aligned}$$

Since Γ_λ is compact and $R_A(z)$ is continuous on Γ_λ, the quantity

$$M = \sup\{\|R(z)\| : z\in\Gamma_\lambda\}$$

is finite.

Thus we obtain

$$r(D_\lambda) = \lim_{k\to\infty}\|D_\lambda^k\|^{1/k} \leq \lim_{k\to\infty}\rho^{1+1/k}M^{1/k} = \rho.$$

Since $\rho > 0$ can be arbitrarily small we arrive at $r(D_\lambda) = 0$. □

Example (in lieu of 12.6.2). We verify $D_\lambda = (A-\lambda I)P_\lambda$ for

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

and $\lambda = 2$. Recall that

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the verification we have

$$(A-2I)P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D_2.$$

Remark. Recall that a matrix $B \in M_n(\mathbb{C})$ is called nilpotent if there is $l \in \mathbb{N}$ such that $B^l = 0$.

Lemma 12.6.3. A matrix $B \in M_n(\mathbb{C})$ satisfies $r(B) = 0$ if and only if B is nilpotent.

Proof. Suppose that $r(B) = 0$.

Then $\sigma(B) = \{0\}$, i.e., every eigenvalue of B is zero.

Hence the characteristic polynomial of B is $p(z) = z^n$.

By the Cayley-Hamilton Theorem, we have have $B^n = p(B) = 0$, which says that B is nilpotent.

Now suppose that B is nilpotent.

Then there exists $l \in \mathbb{N}$ such that $B^l = 0$.

This implies that $B^k = 0$ for all $k \geq l$.

Hence for any matrix norm $\|\cdot\|$ we have $\|B^k\| = 0$ for all $k \geq l$.

This implies that $r(B) = \lim_{k\to\infty} \|B^k\|^{1/k} = 0$. $\square$.

Remark. Lemmas 12.6.1 and 12.6.3 show that the linear operator D_λ is nilpotent.

Definition. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, the nilpotent linear operator D_λ is called the eigennilpotent of A associated with the eigenvalue λ.

Remark 12.6.4. Recall that the order of a nilpotent $B \in M_n(\mathbb{C})$ is the smallest $l \in \mathbb{N}$ such that $B^l = 0$. Since $B^l = B^{l+1} = 0$, then $\mathscr{N}(B^l) = \mathscr{N}(B^{l+1})$, and so $\operatorname{ind}(B) = l$, i.e., the order of B is the same as the index of B. From Exercise 12.6 we have that the index of B is no bigger than n, meaning that $l \leq n$.

Proposition 12.6.5. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, the order m_λ of the eigennilpotent D_λ of A satisfies

$$m_\lambda \leq \dim(\mathscr{R}(P_\lambda)).$$

Proof. It suffices to show that $\mathscr{R}(P_\lambda)$ is D_λ-invariant and that $\mathscr{R}(D_\lambda) \subset \mathscr{R}(P_\lambda)$.

By Lemma 12.5.5 part (iii) we have that $D_\lambda = P_\lambda D_\lambda = D_\lambda P_\lambda$.

To show that $\mathscr{R}(P_\lambda)$ is D_λ-invariant, let $\mathrm{y} \in \mathscr{R}(P_\lambda)$.

Then there exists $\mathrm{x} \in \mathbb{C}^n$ such that $\mathrm{y} = P_\lambda \mathrm{x}$.

Hence $D_\lambda \mathrm{y} = D_\lambda P_\lambda \mathrm{x} = P_\lambda(D_\lambda \mathrm{x}) \in \mathscr{R}(P_\lambda)$, implying $\mathscr{R}(P_\lambda)$ is D_λ-invariant.

To show that $\mathscr{R}(D_\lambda) \subset \mathscr{R}(P_\lambda)$, let $\mathrm{y} \in \mathscr{R}(D_\lambda)$.

Then there is $\mathrm{x} \in \mathbb{C}^n$ such that $\mathrm{y} = D_\lambda \mathrm{x}$.

Hence $\mathrm{y} = D_\lambda \mathrm{x} = P_\lambda(D_\lambda \mathrm{x}) \in \mathscr{R}(P_\lambda)$, implying that $\mathscr{R}(D_\lambda) \subset \mathscr{R}(P_\lambda)$.

That $m_\lambda \leq \dim(\mathscr{R}(P_\lambda))$ follows from Exercise 12.6. □

Remark 12.6.6. Proposition 12.6.5 implies that the resolvent $R_A(z)$ has no essential singularities, so that it is meromorphic on $\rho(A)$. More precisely, part (iv) of Lemma 12.5.5 simplifies to

$$R_A(z) = \frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}} + \sum_{k=0}^{\infty} (-1)^k (z-\lambda)^k S_\lambda^{k+1}$$

and part (v) of Lemma 12.5.5 simplifies to

$$P_\lambda R_A(z) = \frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}}.$$

Remark. We now turn attention to showing that $\mathscr{R}(P_\lambda)$ is the generalized eigenspace $\mathscr{E}_\lambda$, and developing some results to be used in the next section to establish uniqueness of the spectral decomposition. We notice that if $\mathrm{y} \in \mathscr{R}(P_\lambda)$, then $(\lambda I - A)\mathrm{y} \in \mathscr{R}(P_\lambda)$ because for $\mathrm{y} = P_\lambda \mathrm{x}$ we have

$$(\lambda I - A)\mathrm{y} = (\lambda I - A) P_\lambda \mathrm{x} = D_\lambda \mathrm{x},$$

and in the proof of Proposition 12.6.5 we showed that $\mathscr{R}(D_\lambda) \subset \mathscr{R}(P_\lambda)$, whence that $(\lambda I - A)\mathrm{y} \in \mathscr{R}(P_\lambda)$. The converse is also true as we show next.

Lemma 12.6.7. For $A \in M_n(\mathbb{C})$, let $\lambda \in \sigma(A)$ and $\mathrm{y} \in \mathbb{C}^n$. If $(\lambda I - A)\mathrm{y} \in \mathscr{R}(P_\lambda)$, then $\mathrm{y} \in \mathscr{R}(P_\lambda)$.

Proof. Suppose $(\lambda I - A)\mathrm{y} \in \mathscr{R}(P_\lambda)$.

There is nothing to show if $\mathrm{y} = 0$, so assume that $\mathrm{y} \neq 0$, and set $\mathrm{v} = (\lambda I - A)\mathrm{y}$.

Then $\mathrm{v} \in \mathscr{R}(P_\lambda)$ so that $P_\lambda(\mathrm{v}) = \mathrm{v}$ by Lemma 12.1.3.

Independence of the projections means that $P_\mu \mathrm{v} = 0$ for all $\mu \in \sigma(A) \setminus \{\lambda\}$.

As shown in the proof of Lemma 12.6.1 we have $P_\mu A = \mu P_\mu + D_\mu$ for all $\mu \in \sigma(A)$.

Thus for $\mu \in \sigma(A) \setminus \{\lambda\}$ we have

$$0 = P_\mu \mathrm{v} = P_\mu(\lambda I - A)\mathrm{y} = \lambda P_\mu \mathrm{y} - \mu P_\mu \mathrm{y} - D_\mu \mathrm{y} = (\lambda - \mu) P_\mu \mathrm{y} - D_\mu \mathrm{y}.$$

This implies, since $D_\mu P_\mu = D_\mu$ from Lemma 12.5.5 part (iii), that

$$D_\mu(P_\mu \mathrm{y}) = D_\mu \mathrm{y} = (\lambda - \mu)(P_\mu \mathrm{y}).$$

If $P_\mu \mathrm{y} \neq 0$, then $D_\mu^k(P_\mu \mathrm{y}) = (\lambda - \mu)^k (P_\mu \mathrm{y})$ for all $k \in \mathbb{N}$, which would imply that D_μ is not nilpotent since $\mu \neq \lambda$.

But D_μ is nilpotent, so it must be that $P_\mu \mathrm{y} = 0$ for all $\mu \in \sigma(A) \setminus \{\lambda\}$.

From the completeness of the projections $\sum_{\mu \in \sigma(A)} P_\mu = I$, we obtain

$$\mathrm{y} = \sum_{\mu \in \sigma(A)} P_\mu \mathrm{y} = P_\lambda \mathrm{y}.$$

By Lemma 12.1.3 we have that $\mathrm{y} \in \mathscr{R}(P_\lambda)$. $\square$.

Remark 12.6.8. The proof of Lemma 12.6.7 only depended on projections P_μ and nilpotents D_μ satisfying the properties (1) $\sum P_\mu = I$, (2) $P_\mu P_{\mu'} = 0$ for $\mu \neq \mu'$, (3) $D_\mu P_\mu = D_\mu$, and (4) $AP_\mu = \mu P_\mu + D\mu$ for all $\mu \in \sigma(A)$.

Theorem 12.6.9. For $A \in M_n(\mathbb{C})$ and $\lambda \in \sigma(A)$, the generalized eigenspace $\mathscr{E}_\lambda$ is precisely $\mathscr{R}(P_\lambda)$.

Proof. First we show that $\mathscr{E}_\lambda \subset \mathscr{R}(P_\lambda)$.

Recall that $\mathscr{E}_\lambda = \mathscr{N}((\lambda I - A)^{k_\lambda})$ for $k_\lambda = \operatorname{ind}(\lambda I - A)$.

Let $\mathrm{y} \in \mathscr{N}((\lambda I - A)^{k_\lambda})$. Then

$$(\lambda I - A)\big((\lambda I - A)^{k_\lambda - 1}\mathrm{y}\big) = (\lambda I - A)^{k_\lambda}\mathrm{y} = 0 \in \mathscr{R}(P_\lambda).$$

By Lemma 12.6.7 we have that $(\lambda I - A)^{k_\lambda - 1}\mathrm{y} \in \mathscr{R}(P_\lambda)$.

Then

$$(\lambda I - A)\big((\lambda I - A)^{k_\lambda - 2}\mathrm{y}\big) = (\lambda I - A)^{k_\lambda - 1}\mathrm{y} \in \mathscr{R}(P_\lambda),$$

so by Lemma 12.6.7 we have that $(\lambda I - A)^{k_\lambda - 2}\mathrm{y} \in \mathscr{R}(P_\lambda)$.

We continue repeating this argument until we obtain $(\lambda I - A)\mathrm{y} \in \mathscr{R}(P_\lambda)$, which implies by Lemma 12.6.7 that $\mathrm{y} \in \mathscr{R}(P_\lambda)$.

This shows that $\mathscr{E}_\lambda \subset \mathscr{R}(P_\lambda)$.

To get $\mathscr{E}_\lambda = \mathscr{R}(P_\lambda)$ for all $\lambda \in \sigma(A)$, we compare two direct sum decompositions of $\mathbb{C}^n$ both indexed over the spectrum of A.

The first direct sum decomposition is the one from Theorem 12.2.14, namely that

$$\mathbb{C}^n = \sum_{\lambda \in \sigma(A)} \mathscr{E}_\lambda.$$

The second direct sum decomposition is one from follows from the completeness of the spectral projections,

$$I = \sum_{\lambda \in \sigma(A)} P_\lambda.$$

For any $\mathrm{x} \in \mathbb{C}^n$ we have

$$\mathrm{x} = \sum_{\lambda \in \sigma(A)} P_\lambda \mathrm{x} \in \sum_{\lambda \in \sigma(A)} \mathscr{R}(P_\lambda).$$

To show this sum is direct we need that

$$\mathscr{R}(P_\lambda) \cap \left(\sum_{\mu \in \sigma(A) \setminus \{\lambda\}} \mathscr{R}(P_\mu) \right) = \{0\}$$

as required by Definition 1.3.6.

To this end we suppose

$$\mathrm{y} \in \mathscr{R}(P_\lambda) \cap \left(\sum_{\mu \in \sigma(A) \setminus \{\lambda\}} \mathscr{R}(P_\mu) \right).$$

Then $\mathrm{y} \in \mathscr{R}(P_\lambda)$ and there exist w_μ, $\mu \in \sigma(A) \setminus \{\lambda\}$, such that

$$\mathrm{y} = \sum_{\mu \in \sigma(A) \setminus \{\lambda\}} P_\mu w_\mu.$$

Since $\mathrm{y} \in \mathscr{R}(P_\lambda)$, we have that $\mathrm{y} = P_\lambda \mathrm{y}$ by Lemma 12.1.3.

Since $P_\lambda P_\mu = 0$ for all $\mu \in \sigma(A) \setminus \{\lambda\}$ by Theorem 12.4.2, we have that

$$\mathrm{y} = P_\lambda \mathrm{y} = \sum_{\mu \in \sigma(A) \setminus \{\lambda\}} P_\lambda P_\mu (w_\mu) = 0.$$

This gives the direct sum decomposition

$$\mathbb{C}^n = \sum_{\lambda \in \sigma(A)} \mathscr{R}(P_\lambda).$$

The inclusion $\mathscr{E}_\lambda \subset \mathscr{R}(P_\lambda)$ for all $\lambda \in \sigma(A)$ forces the direct summands of the two direct sum decompositions of $\mathbb{C}^n$ to be the same, namely $\mathscr{E}_\lambda = \mathscr{R}(P_\lambda)$ for all $\lambda \in \sigma(A)$. $\square$

Remark 12.6.10. The proof of Theorem 12.6.9 only depends on projections with the properties listed in Remark 12.6.8. This is important in the next section when we prove the uniqueness of the spectral decomposition.

Theorem 12.6.12 (Spectral Decomposition Theorem). For $A \in M_n(\mathbb{C})$, and $\lambda \in \sigma(A)$, let P_λ be the spectral projection of A associated to λ, and let D_λ be the eigennilpotent of A associated to λ with its order m_λ. The resolvent of A takes the form

$$R_A(z) = \sum_{\lambda \in \sigma(A)} \left[\frac{P_\lambda}{z - \lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-k)^{k+1}} \right],$$

and there holds the spectral decomposition

$$A = \sum_{\lambda \in \sigma(A)} \big(\lambda P_\lambda + D\lambda \big).$$

Proof. From Lemma 12.5.5 part (v) and the nilpotency of D_λ we have

$$R_A(z)P_\lambda = \frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}}.$$

Combining this with the completeness of the spectral projections gives

$$\begin{aligned} R_A(z) &= R_A(z)I \\ &= R_A(z) \sum_{\lambda\in\sigma(A)} P_\lambda \\ &= \sum_{\lambda\in\sigma(A)} R_A(z)P_\lambda \\ &= \sum_{\lambda\in\sigma(A)} \left[\frac{P_\lambda}{z-\lambda} + \sum_{k=1}^{m_\lambda - 1} \frac{D_\lambda^k}{(z-\lambda)^{k+1}} \right]. \end{aligned}$$

This is the stated form of the resolvent $R_A(z)$.

We saw in the proof of Lemma 12.6.1 that $AP\lambda = \lambda P_\lambda + D_\lambda$.

Combining this with the completeness of the spectral projections gives

$$A = AI = A \sum_{\lambda\in\sigma(A)} P_\lambda = \sum_{\lambda\in\sigma(A)} AP_\lambda = \sum_{\lambda\in\sigma(A)} \big(\lambda P_\lambda + D_\lambda\big).$$

This is the stated spectral projection of A. $\square$

Remark. The form of the resolvent stated in the Spectral Decomposition Theorem is the precisely form we have already been getting by using the partial fraction decompositions for the rational function entries of the resolvent.

Example (in lieu of 12.6.13). Find the spectral decomposition for the linear operator

$$A = \begin{bmatrix} -1 & 11 & -3 \\ -2 & 8 & -1 \\ -1 & 5 & 0 \end{bmatrix}.$$

The characteristic polynomial of A is

$$\det(zI - A) = z^3 - 7z^2 + 16z - 12 = (z-2)^2(z-3).$$

The adjugate of $zI - A$ is

$$\operatorname{adj}(zI - A) = \begin{bmatrix} z^2 - 8z + 5 & 11z - 15 & -3z + 13 \\ -2z + 1 & z^2 + z - 3 & -z + 5 \\ -z - 2 & 5z - 6 & z^2 - 7z + 14 \end{bmatrix}.$$

Performing nine partial fraction decompositions (one for each entry of the resolvent) gives

$$R_A(z) = \frac{1}{z-2}\begin{bmatrix} 11 & -18 & -4 \\ 5 & -8 & -2 \\ 5 & -9 & -1 \end{bmatrix} + \frac{1}{(z-2)^2}\begin{bmatrix} 7 & -7 & -7 \\ 3 & -3 & -3 \\ 4 & -4 & -4 \end{bmatrix} + \frac{1}{z-3}\begin{bmatrix} -10 & 18 & 4 \\ -5 & 9 & 2 \\ -5 & 9 & 2 \end{bmatrix}.$$

This form of the resolvent is the one stated in the Spectral Decomposition Theorem.

From this form of the resolvent we have

$$P_2 = \begin{bmatrix} 11 & -18 & -4 \\ 5 & -8 & -2 \\ 5 & -9 & -1 \end{bmatrix}, \ D_2 = \begin{bmatrix} 7 & -7 & -7 \\ 3 & -3 & -3 \\ 4 & -4 & -4 \end{bmatrix}, \ P_3 = \begin{bmatrix} -10 & 18 & 4 \\ -5 & 9 & 2 \\ -5 & 9 & 2 \end{bmatrix}.$$

The spectral decomposition is

$$A = 2P_2 + D_2 + 3P_3.$$

One purpose for having the spectral decomposition of A is in finding quicker means of computing powers of A in terms of spectral decompositions, such as

$$\begin{aligned} A^2 &= (2P_2 + D_2 + 3P_3)(2P_2 + D_2 + 3P_3) \\ &= 4P_2^2 + 2P_2D_2 + 6P_2P_3 + 2D_2P_2 + D_2^2 + 3D_2P_3 + 6P_3P_2 + 3P_3D_2 + 9P_3^2 \\ &= 4P_2 + 4D_2 + 9P_3. \end{aligned}$$

Not only can we take powers of A we can also take holomorphic images of A, and get expressions that look an awful lot like spectral decompositions!

Corollary 12.6.14. For $A \in M_n(\mathbb{C})$, let f be holomorphic complex valued function defined on a simply connected open set containing $\sigma(A)$. If for $\lambda \in \sigma(A)$, the complex constants $a_{n,\lambda}$ are the coefficients in the power series expansion of f about λ, i.e.,

$$f(z) = f(\lambda) + \sum_{n=1}^{\infty} a_{n,\lambda}(z-\lambda)^n,$$

then

$$f(A) = \sum_{\lambda \in \sigma(A)} \left[f(\lambda)P_\lambda + \sum_{k=1}^{m_\lambda - 1} a_{k,\lambda} D_\lambda^k \right].$$

In the case that A is semisimple the expression simplifies to

$$f(A) = \sum_{\lambda \in \sigma(A)} f(\lambda)P_\lambda.$$

Proof. For each $\lambda \in \sigma(A)$ let Γ_λ be a small circle lying in the simply open set on which f is holomorphic, and also enclosing λ but no other element of $\sigma(A)$.

Set $a_{0,\lambda} = f(\lambda)$.

By the Spectral Resolution Theorem, interchanging of integration and uniform convergence of series, we have

$$\begin{aligned} f(A) &= \frac{1}{2\pi i} \sum_{\lambda \in \sigma(A)} \oint_{\Gamma_\lambda} f(z) R_A(z)\ dz \\ &= \frac{1}{2\pi i} \sum_{\lambda \in \sigma(A)} \oint_{\Gamma_\lambda} \sum_{k=0}^{\infty} a_{k,\lambda}(z-\lambda)^k R_A(z)\ dz \\ &= \frac{1}{2\pi i} \sum_{\lambda \in \sigma(A)} \sum_{k=0}^{\infty} a_{k,\lambda} \oint_{\Gamma_\lambda} (z-\lambda)^k R_A(z)\ dz. \end{aligned}$$

By the Spectral Decomposition Theorem we have that

$$R_A(z) = \sum_{\lambda \in \sigma(A)} \left[\frac{P_\lambda}{z - \lambda} + \sum_{l=1}^{m_\lambda - 1} \frac{D_\lambda^l}{(z-\lambda)^{l+1}} \right].$$

Thus

$$\oint_{\Gamma_\lambda} (z-\lambda)^k R_A(z)\ dz = \begin{cases} P_\lambda & \text{if } \ k = 0, \\ D_\lambda^k & \text{if } \ k = 1, \dots, m_\lambda - 1, \\ 0 & \text{if } \ k \geq m_\lambda. \end{cases}$$

All of the terms with $k \geq m_\lambda$ in the power series vanish, leaving the finite sum

$$f(A) = \sum_{\lambda \in \sigma(A)} \left[f(\lambda) P_\lambda + \sum_{k=1}^{m_\lambda - 1} a_{k,\lambda} D_\lambda^k \right].$$

This gives the result. □

Example (in lieu of 12.6.15). In the previous example, we used the spectral decomposition

$$A = 2P_2 + D_2 + 3P_3$$

to compute

$$A^2 = 4P_2 + 4D_2 + 9P_3.$$

We will use Corollary 12.6.14 to compute this by finding the coefficients of the power series expansion of the square function expanded about $\lambda = 2$:

$$f(z) = z^2 = (z - 2 + 2)^2 = ((z-2) + 2)^2 = 4 + 4(z-2) + (z-2)^2.$$

The Taylor series coefficients of $f(z)$ about $\lambda = 2$ are

$$a_{0,2} = 4,\ a_{1,2} = 4,\ a_{2,2} = 1,\ a_{k,2} = 0 \text{ for all } k \geq 3.$$

Since $D_3 = 0$ we do not need the Taylor coefficients of $f(z) = z^2$ expanded about $\lambda = 3$. By Corollary 12.6.14, using $a_{1,2} = 4$, we have

$$A^2 = f(A) = \sum_{\lambda \in \sigma(A)} \left[f(\lambda) P_\lambda + \sum_{k=1}^{m_\lambda - 1} a_{k,\lambda} D_\lambda^k \right] = 2^2 P_2 + 4D_2 + 3^2 P_3.$$

This agrees with what we computed earlier.