Math 346 Lecture #37
12.7 Spectral Mapping Theorem

Recall the Semisimple Spectral Mapping Theorem 4.3.12 which states that for a semisimple $A \in M_n(\mathbb{C})$ and a polynomial $p \in \mathbb{C}[x]$, the set of eigenvalues of the the linear operator $p(A)$ is precisely $\{p(\lambda) : \lambda \in \sigma(A)\}$. We extend this result in two ways: to all linear operators $A \in M_n(\mathbb{C})$ and to all complex-valued functions holomorphic on a simply connected open set containing the spectrum of a given linear operator.

Additionally we prove the uniqueness of the spectral decomposition of a linear operator. This shows that the conclusion of Corollary 12.6.14 is the spectral decomposition of $f(A)$. Finally we use the spectral decomposition theory to develop the power method, a means of computing the eigenvector of a linear operator that has a dominant eigenvalue.

12.7.1 The Spectral Mapping Theorem

For $A \in M_n(\mathbb{C})$ and f holomorphic on an open simply connected set containing $\sigma(A)$ we have two ways to represent the linear operator $f(A) \in M_n(\mathbb{C})$, one by contour integral (the Spectral Resolution formula), and another by means of the spectral decomposition of A and coefficients of the power series of f about the eigenvalues of A (Corollary 12.6.14).

Theorem 12.7.1 (Spectral Mapping Theorem). For $A \in M_n(\mathbb{C})$, if f is holomorphic on an open disk containing $\sigma(A)$, then

$$\sigma(f(A)) = f(\sigma(A)).$$

Moreover, if $\mathrm{x} \in \mathbb{C}^n$ is an eigenvector of A corresponding to $\lambda \in \sigma(A)$, then x is an eigenvector of $f(A)$ corresponding to $f(\lambda)$.

Proof. We have equality of two sets $f(\sigma(A))$ and $\sigma(f(A))$ to show.

For the inclusion $\sigma(f(A)) \subset f(\sigma(A))$ we show that $\mu \notin f(\sigma(A))$ implies $\mu \notin \sigma(f(A))$.

For $\mu \notin f(\sigma(A))$, the function $z \to f(z) - \mu$ is both holomorphic in a neighbourhood of $\sigma(A)$ and is nonzero on $\sigma(A)$.

Hence $f(A) - \mu I$ is nonsingular, meaning that $\mu \notin \sigma(f(A))$.

For the inclusion $f(\sigma(A)) \subset \sigma(f(A))$ we start with $\mu \in f(\sigma(A))$.

Then there exists $\lambda \in \sigma(A)$ such that $\mu = f(\lambda)$.

The holomorphic function f is defined on open disk U that contains $\sigma(A)$.

Define a function g on U by

$$g(z) = \begin{cases} \dfrac{f(z) - f(\lambda)}{z - \lambda} & \text{if } z \neq \lambda, \\ f'(\lambda) & \text{if } z = \lambda. \end{cases}$$

This function g is holomorphic on the punctured neighbourhood $U \setminus \{\lambda\}$ and it is continuous at λ.

By Exercise 11.19, the function g is holomorphic on U and satisfies

$$g(z)(z - \lambda) = f(z) - f(\lambda) = f(z) - \mu \text{ for all } z \in U.$$

Because $g(z)(z-\lambda)$ and $f(z)-\mu$ are holomorphic on U we have by the Spectral Resolution formula that, for a simply closed positively oriented contour Γ enclosing $\sigma(A)$, there holds

$$\begin{aligned} g(A)(A-\lambda I) &= \frac{1}{2\pi i}\oint_\Gamma g(z)(z-\lambda)R_A(z)\ dz \\ &= \frac{1}{2\pi i}\oint_\Gamma (f(z)-\mu)R_A(z)\ dz \\ &= f(A)-\mu I. \end{aligned}$$

Let x be an eigenvector of A corresponding to the eigenvector λ.

Then we have that

$$(f(A)-\mu I)\mathrm{x} = g(A)(A-\lambda I)\mathrm{x} = g(A)0 = 0.$$

This means $f(A)-\mu I$ is singular, so that $\mu \in \sigma(f(A))$ with x an eigenvector of $f(A)$ corresponding to $\mu \in \sigma(f(A))$. □

Example (in lieu of 12.7.2). Recall that the eigenvalue/eigenvector pairs of

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

are $\lambda_1 = 3$, $\xi_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}$ and $\lambda_2 = -1$, $\xi_2 = \begin{bmatrix} 1 & -2 \end{bmatrix}^{\mathrm{T}}$.

The solution of the initial value problem $\mathrm{x}' = A\mathrm{x}$, $\mathrm{x}(0) = \mathrm{x}_0$ is

$$\mathrm{x}(t) = \exp(tA)\mathrm{x}_0.$$

For each constant $\alpha \in \mathbb{C}$, the function $f_\alpha(z) = e^{\alpha z}$ is entire.

By the Spectral Mapping Theorem, we have

$$\sigma(f_\alpha(A)) = f_\alpha(\sigma(A)) = \{e^{\alpha\lambda_1}, e^{\alpha\lambda_2}\} = \{e^{3\alpha}, e^{-\alpha}\},$$

and the eigenvectors ξ_1, ξ_2 of A are eigenvectors of $f_\alpha(A) = \exp(\alpha A)$ corresponding to $e^{3\alpha}$ and $e^{-\alpha}$ respectively.

Restricting $\alpha \in \mathbb{R}$ we obtain the eigenvalues e^{3t} and e^{-t} with their corresponding eigenvectors ξ_1 and ξ_2 for $\exp(tA)$.

In particular, since $\exp(tA)\xi_2 = e^{-t}\xi_2$ for each $t \geq 0$, we have

$$\lim_{t\to\infty} \exp(tA)\xi_2 = \lim_{t\to\infty} e^{-t}\xi_2 = 0.$$

Similarly we would get

$$\lim_{t\to-\infty} \exp(tA)\xi_1 = \lim_{t\to-\infty} e^{3t}\xi_1 = 0.$$

The point of all of this is that we can compute these limits by means of the Spectral Mapping Theorem without explicitly computing $\exp(tA)$.

12.7.2 Uniqueness of the Spectral Decomposition

We now show that uniqueness of the spectral decomposition of a linear operator. We do this by supposing there is a collection of projections and nilpotents with the properties by which they form a spectral decomposition for a given linear operator, and then shown that the collection of projections and nilpotents are indeed the spectral projections and eigennilpotents of that linear operator. We make use of Lemma 12.6.7, Remark 12.6.8, Theorem 12.6.9, and Remark 12.6.10 wherein we noted that the proofs of Lemma 12.6.7 and Theorem 12.6.9 only depended on a collection of projections and nilpotents satisfying certain properties.

Theorem 12.7.5. For $A \in M_n(\mathbb{C})$, if for each $\lambda \in \sigma(A)$ there is a projection $Q_\lambda \in M_n(\mathbb{C})$ and a nilpotent $C_\lambda \in M_n(\mathbb{C})$ satisfying

(i) $Q_\lambda Q_\mu = 0$ for all $\mu \in \sigma(A)$ with $\lambda \neq \mu$,
(ii) $Q_\lambda C_\lambda = C_\lambda Q_\lambda = C_\lambda$,
(iii) $Q_\mu C_\lambda = C_\lambda Q_\mu = 0$ for all $\mu \in \sigma(A)$ with $\mu \neq \lambda$,
(iv) $\sum_{\lambda\in\sigma(A)} Q_\lambda = I$, and
(v) $A = \sum_{\lambda\in\sigma(A)} \big(\lambda Q_\lambda + C_\lambda\big)$

then for each $\lambda \in \sigma(A)$ the projection Q_λ is the eigenprojection P_λ associated to A and the nilpotent C_λ is the eigennipotent D_λ associated to A.

Proof. For every $\mu \in \sigma(A)$ we have by item (v), the "spectral decomposition" of A, and items (i), (ii), and (iii) that

$$\begin{aligned} AQ_\mu &= \left(\sum_{\lambda\in\sigma(A)} \big(\lambda Q_\lambda + C_\lambda\big)\right) Q_\mu \\ &= \sum_{\lambda\in\sigma(A)} \big(\lambda Q_\lambda Q_\mu + C_\lambda Q_\mu\big) \\ &= \mu Q_\mu^2 + C_\mu Q_\mu \\ &= \mu Q_\mu + C_\mu \end{aligned}$$

This implies that

$$C_\mu = (A - \mu I)Q_\mu.$$

Since $D_\mu = (A - \mu I)P_\mu$ by Lemma 12.6.1, it suffices to show that $P_\mu = Q_\mu$ for all $\mu \in \sigma(A)$.

Using again item (v), the "spectral decomposition" and items (i), (ii), and (iii) we obtain

$$Q_\mu A = Q_\mu \sum_{\lambda\in\sigma(A)} \big(\lambda Q_\lambda + C_\lambda\big) = \lambda Q_\mu + C_\mu.$$

This is the situation we have in the proof of Lemma 12.6.7, and so we obtain the conclusion of Lemma 12.6.7, namely that if $(\mu I - A)\mathrm{y} \in \mathscr{R}(Q_\mu)$ then $\mathrm{y} \in \mathscr{R}(Q_\mu)$.

Now we can adapt the proof of Theorem 12.6.9 to show that $\mathscr{E}_\mu = \mathscr{R}(Q_\mu)$, and this holds for every $\mu \in \sigma(A)$.

For $\mathrm{v} \in \mathbb{C}^n$ and $\lambda \in \sigma(A)$ we have $Q_\lambda \mathrm{v} \in \mathscr{E}_\lambda = \mathscr{R}(P_\lambda)$.

For $\mu \in \sigma(A) \setminus \{\lambda\}$ we have $P_\mu P_\lambda = 0$ so that $P_\mu Q_\lambda \mathrm{v} = 0$ since $Q_\lambda \mathrm{v} \in \mathscr{R}(P_\lambda)$.

Since $\mathscr{R}(Q_\mu) = \mathscr{E}_\mu$ and P_μ is a projection with the same range as Q_μ we have that $P_\mu Q_\mu \mathrm{v} = Q_\mu \mathrm{v}$ for every $\mathrm{v} \in \mathbb{C}^n$ by Lemma 12.1.3.

Using item (iv), the completeness of the projections Q_λ, $\lambda \in \sigma(A)$, we have for a fixed $\mu \in \sigma(A)$ that

$$P_\mu \mathrm{v} = P_\mu I \mathrm{v} = P_\mu \sum_{\lambda \in \sigma(A)} Q_\lambda \mathrm{v} = \sum_{\lambda \in \sigma(A)} P_\mu Q_\lambda \mathrm{v} = P_\mu Q_\mu \mathrm{v} = Q_\mu \mathrm{v}.$$

This implies that $P_\mu = Q_\mu$. □

Theorem 12.7.6 (Mapping the Spectral Decomposition). Let $A \in M_n(\mathbb{C})$ and f be holomorphic on a simply connected open set U containing $\sigma(A)$. If for each $\lambda \in \sigma(A)$ we have the Taylor series

$$f(z) = f(\lambda) + \sum_{k=1}^{\infty} a_{n,\lambda}(z-\lambda)^k,$$

then

$$f(A) = \sum_{\lambda \in \sigma(A)} \left(f(\lambda) P_\lambda + \sum_{k=1}^{m_\lambda - 1} a_{k,\lambda} D_\lambda^k \right)$$

is the spectral decomposition of $f(A)$, i.e., for each $\nu \in \sigma(f(A))$ the eigenprojection for $f(A)$ is given by

$$\sum_{\mu \in \sigma(A), f(\mu) = \nu} P_\mu,$$

and the corresponding eigennilpotent D_ν is given by

$$\sum_{\mu \in \sigma(A), f(\mu) = \nu} \sum_{k=1}^{m_\mu - 1} a_{k,\mu} D_\mu^k.$$

Sketch of the Proof. By the Spectral Mapping Theorem, the spectrum of $f(A)$ is $f(\sigma(A))$.

Corollary 12.6.14 gives a spectral decomposition

$$f(A) = \sum_{\lambda \in \sigma(A)} \left(f(\lambda) P_\lambda + \sum_{k=1}^{m_\lambda - 1} a_{k,\lambda} D_\lambda^k \right).$$

Forming the projections and the nilpotents above one then shows that these satisfy the requirements of Theorem 12.7.5.

Hence these are the unique eigenprojections and eigennilpotents of $f(A)$. □

Example (in lieu of 12.7.7). Find the eigenprojections and eigennilpotents of the square of

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

using the formulas in Theorem 12.7.6.

From the partial fraction decomposition of the entries of the resolvent $R_A(z)$ we obtain

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/9 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The spectrum of A is $\sigma(A) = \{1, -1, 2\}$ and the spectral decomposition is

$$A = P_1 + D_1 - P_{-1} + D_{-1} + 2P_2.$$

Since $f(z) = z^2$ is entire, we have by the Spectral Mapping Theorem that

$$\sigma(f(A)) = f(\sigma(A)) = \{1^2, (-1)^2, 2^2\} = \{1, 4\}.$$

The eigenprojection for $f(A)$ corresponding to $\nu = 1 \in \sigma(f(A))$ is

$$\begin{aligned} \sum_{\mu\in\sigma(A), f(\mu)=1} P_\mu = P_1 + P_{-1} &= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/9 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1/9 \\ 0 & 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The eigenprojection for $f(A)$ corresponding to $\nu = 4 \in \sigma(f(A))$ is P_2.

To get the eigennilpotent for $f(A)$ corresponding to $\nu = 1$ we compute the Taylor series expansions of $f(z) = z^2$ about $z = 1$ and $z = -1$:

$$z^2 = (1 + z - 1)^2 = 1 + 2(z-1) + (z-1)^2,$$
$$z^2 = (-1 + z + 1)^2 = 1 - 2(z+1) + (z+1)^2.$$

Here we have $a_{1,1} = 2$ and $a_{1,-1} = -2$.

The eigennilpotent for $f(A)$ corresponding to $\nu = 1$ is

$$\begin{aligned}\sum_{\mu\in\sigma(A), f(\mu)=1} \sum_{k=1}^{m_\mu - 1} a_{k,\mu} D_\mu^k &= 2D_1 - 2D_{-1} \\ &= \begin{bmatrix} 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.\end{aligned}$$

We can verify all of the eigenprojections and eigennilpotents for $f(A)$ by directly squaring the spectral decomposition of A:

$$\begin{aligned} A^2 &= (P_1 + D_1 - P_{-1} + D_{-1} + 2P_2)(P_1 + D_1 - P_{-1} + D_{-1} + 2P_2) \\ &= P_1 + 2D_1 + P_{-1} - 2D_{-1} + 4P_2 \\ &= P_1 + P_{-1} + 2D_1 - 2D_{-1} + 4P_2. \end{aligned}$$

12.7.3 The Power Method

The power method is algorithm to find an eigenvector for certain type of linear operator on a finite dimensional vector space. The linear operator A must have a dominant semisimple eigenvalue $\lambda \in \sigma(A)$ which means that $|\lambda| > |\mu|$ for all $\mu \in \sigma(A) \setminus \{\lambda\}$ (λ is the dominant eigenvalue) and that the geometric multiplicity of λ is equal to its algebraic multiplicity (λ is a semisimple eigenvalue).

We state the power method when the dominant eigenvalue is one. It is HW (Exercise 12.34) to extend the following result when the dominant eigenvalue is something other than one.

Theorem 12.7.8. For $A \in M_n(\mathbb{C})$, suppose that $1 \in \sigma(A)$ is semisimple and dominant. If $\mathrm{v} \in \mathbb{C}^n$ satisfies $P_1\mathrm{v} \neq 0$, then for any norm $\|\cdot\|$ on $\mathbb{C}^n$ there holds

$$\lim_{k\to\infty} \|A^k\mathrm{v} - P_1\mathrm{v}\| = 0.$$

See the book for the proof.

Remark. The power method consist in making a good initial guess v for an eigenvector corresponding to the dominant semisimple eigenvalue 1. The "good" part of the initial guess is that $P_1\mathrm{v} \neq 0$, because $P_1\mathrm{v} \in \mathscr{E}_1 \setminus \{0\}$ is an eigenvector. Although v is not necessarily an eigenvector, its iterates $A^k\mathrm{v}$ converge in any matrix norm to the eigenvector $P_1\mathrm{v}$ and the rate of convergence is determined by the dominance of the dominant semisimple eigenvalue.

Example (in lieu of 12.7.9). The linear operator

$$A = \begin{bmatrix} 1/4 & 3/4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

is semisimple with spectrum $\sigma(A) = \{1, 1/4\}$ where $1/4$ has algebraic multiplicity 2.

This means that eigenvalue 1 is semisimple and dominant.

To find an eigenvector corresponding to eigenvalue 1 by the power method we start with the guess $\mathrm{v} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$.

Then

$$A\mathrm{v} = \begin{bmatrix} 1 \\ 1 \\ 1/4 \end{bmatrix},\ A^2\mathrm{v} = \begin{bmatrix} 1 \\ 1 \\ 1/16 \end{bmatrix},\ A^3\mathrm{v} = \begin{bmatrix} 1 \\ 1 \\ 1/64 \end{bmatrix}, \ldots, A^k\mathrm{v} = \begin{bmatrix} 1 \\ 1 \\ 1/4^k \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Theorem 12.7.8 says this limit is an eigenvector of A corresponding to the eigenvalue 1 and that it is the image of the eigenprojection of the initial guess.

From the partial fraction decomposition of the resolvent $R_A(z)$ we have

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } P_{1/4} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We verify the spectral decomposition $A = P_1 + (1/4)P_{1/4}$.

The limit of $A^k\mathrm{v}$ is indeed

$$P_1\mathrm{v} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$