Math 346 Lecture #39
12.9 The Drazin Inverse

Construction of pseudo-inverses are consequences of decompositions of noninvertible linear operators. The Moore-Penrose inverse of a noninvertible $A \in M_n(\mathbb{C})$ is

$$A^\dagger = V_1 \Sigma U_1^{\mathrm{H}}$$

that derives from the compact form $A = U_1 \Sigma_1 V_1^{\mathrm{H}}$ of an SVD for A. Recall that the columns of V_1 form a orthonormal basis for $\mathscr{R}(A^{\mathrm{H}})$ and that the columns of U_1 form an orthonormal basis for $\mathscr{R}(A)$. The Moore-Penrose inverse $A^\dagger$ satisfies

$$AA^\dagger = \mathrm{proj}_{\mathscr{R}(A)} \text{ and } A^\dagger A = \mathrm{proj}_{\mathscr{R}(A^{\mathrm{H}})}.$$

These say that $AA^\dagger$ and $A^\dagger A$ are geometrically the nearest possible approximations to identity operator I. Another approach to constructing a pseudo-inverse of a noninvertible linear operator uses its spectral decomposition, and this leads to the Drazin inverse, whose properties are more spectral than geometric.

12.9.1 Definition and Spectral Decomposition

Definition 12.9.1. Suppose $A \in M_n(\mathbb{C})$ is nonzero, noninvertible, and

$$A = \sum_{\lambda \in \sigma(A)} \big(\lambda P_\lambda + D_\lambda\big)$$

is its spectral decomposition, i.e., $0 \in \sigma(A)$ so that $P_0 \neq 0$.

Set $P_* = I - P_0$.

Then P_* is the complementary projection (see Example 12.1.2) for which

$$\mathscr{N}(P_*) = \mathscr{R}(I - P_*) = \mathscr{R}(P_0)$$

by Lemma 12.1.3, i.e., the kernel of P_* is the generalized eigenspace $\mathscr{R}(P_0) = \mathscr{E}_0$ for the eigenvalue 0 of A.

The complementary projection P_* also satisfies

$$\mathbb{C}^n = \mathscr{N}(P_*) \oplus \mathscr{R}(P_*) = \mathscr{R}(P_0) \oplus \mathscr{R}(P_*)$$

by Theorem 12.1.4.

The subspaces $\mathscr{R}(P_0)$ and $\mathscr{R}(P_*)$ are both A-invariant because P_0 and P_* both commute with A, and thus the direct sum decomposition $\mathscr{R}(P_0) \oplus \mathscr{R}(P_*)$ is A-invariant.

Since $I = P_0 + P_*$ we have the Wedderburn Decomposition

$$A = AI = A(P_0 + P_*) = AP_0 + AP_* = (A + 0I)P_0 + AP_* = D_0 + AP_*,$$

where we have used $D_\lambda = (A - \lambda I)P_\lambda$ for $\lambda = 0$ from Lemma 12.6.1.

Definition 12.9.2. Continuing with the notation established above, let C be the restriction of A to the A-invariant nontrivial proper subspace $\mathscr{R}(P_*)$.

The operator C has no zero eigenvalue. [If there is $\mathrm{x} \in \mathscr{R}(P_*)$ such that $A\mathrm{x} = 0$ then $\mathrm{x} \in \mathscr{E}_0 = \mathscr{R}(P_0)$, which implies since $\mathbb{C}^n = \mathscr{R}(P_0) \oplus \mathscr{R}(P_*)$ that $\mathrm{x} = 0$, whence that C has no zero eigenvalue.]

Thus the operator C is injective (having trivial kernel), and hence invertible.

The Drazin inverse of nonzero, noninvertible $A \in M_n(\mathbb{C}^n)$ is the linear operator A^D from $\mathbb{C}^n$ to $\mathscr{R}(P_*)$ defined by

$$A^D = C^{-1}P_*.$$

Alternatively, by uniquely writing $\mathrm{x} = \mathrm{x}_0 + \mathrm{x}_* \in \mathscr{R}(P_0) \oplus \mathscr{R}(P_*)$ we can also define

$$A^D\mathrm{x} = C^{-1}\mathrm{x}_*.$$

Because C is the restriction of A to $\mathscr{R}(P_*)$, we have that $AA^D\mathrm{x} = AC^{-1}P_*\mathrm{x} = P_*\mathrm{x}$, which is to say that

$$AA^D = P_*.$$

On the other hand, since A and P_* commute, we have that $A^DA\mathrm{x} = C^{-1}P_*A\mathrm{x} = C^{-1}AP_*\mathrm{x} = P_*\mathrm{x}$ which is to say that

$$A^DA = P_*.$$

This establishes part (i) of Proposition 12.9.9, that $AA^D = A^DA$, without having to represent A in a block diagonal manner as the book does, and it gives a means to verify a computed A^D by checking if AA^D and A^DA agree and equal $P_* = I - P_0$.

Remark 12.9.3. When A is invertible, then $P_* = I$ and $C = A$ so that $A^D = C^{-1} \circ P_* = A^{-1}$. When $A = 0$ then $P_0 = I$ so that $P_* = I - P_0 = 0$, whence $A^D = 0$.

Remark 12.9.4. Because $\mathbb{C}^n = \mathscr{R}(P_*) \oplus \mathscr{R}(P_0)$ there exists an invertible $S \in M_n(\mathbb{C})$ such that

$$A = S^{-1}\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} S$$

where (without the book providing justification) we have

$$AP_* = S^{-1}\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} S, \ D_0 = S^{-1}\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} S, \text{ and } A^D = S^{-1}\begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} S.$$

You have HW (Exercise 12.43) using these formulas and the Wedderburn decomposition to compute A^D for the matrix in Example 12.9.6.

Theorem 12.9.5. For $A \in M_n(\mathbb{C})$ with spectral decomposition

$$A = \sum_{\lambda \in \sigma(A)} \big(\lambda P_\lambda + D_\lambda\big)$$

the Drazin inverse of A has the spectral decomposition

$$A^D = \sum_{\lambda \in \sigma(A)\setminus\{0\}} \left(\frac{P_\lambda}{\lambda} + \sum_{\ell=1}^{m_\lambda - 1} \frac{(-1)^\ell D_\lambda^\ell}{\lambda^{\ell+1}} \right)$$

where m_λ are the algebraic multiplicities of the nonzero eigenvalues $\lambda \in \sigma(A)$. [The formula in the book for A^D is missing the power ℓ on the eigennipotent D_λ.]

See book for the proof.

Remark. Two additional properties of the Drazin inverse are stated in Proposition 12.9.9: (ii) if $\operatorname{ind}(A) = k$, then $A^{k+1}A^D = A^k$ and (iii) $A^D A A^D = A^D$. Property (ii) implies that $A^{m+1}A^D = A^m$ for all $m \geq \operatorname{ind}(A)$ by induction.

Example. Compute and compare the Moore-Penrose inverse $A^\dagger$ and Drazin inverse A^D for the nonzero, noninvertible

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the compact SVD

$$A = \mathrm{U}_1 \Sigma_1 V_1^{\mathrm{T}} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \\ 0 & 0 \end{bmatrix}^{\mathrm{H}}$$

we compute the Moore-Penrose inverse

$$\begin{aligned} A^\dagger &= V_1 \Sigma_1^{-1} U_1^{\mathrm{H}} \\ &= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \\ 0 & 0 \end{bmatrix}^{\mathrm{H}} \\ &= \begin{bmatrix} 1/2 & -3/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Here

$$AA^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^\dagger A$$

as is expected since $\mathscr{R}(A)$ and $\mathscr{R}(A^{\mathrm{H}})$ are the same (the columns of U_1 and V_1 span the same subspace).

The from the partial fraction decomposition of the resolvent

$$R_A(z) = \frac{1}{z-2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{(z-2)^3} \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{z-0} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we identity the eigenprojections and eigennilpotents,

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; D_2 = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; P_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and hence the spectral decomposition

$$A = 2P_2 + D_2 + 0P_0.$$

By Theorem 12.9.5 the Drazin Inverse of A is

$$A^D = \frac{1}{2}P_2 - \frac{1}{2^2}D_2 = \begin{bmatrix} 1/2 & -3/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Drazin inverse agrees with the Moore-Penrose inverse, and we have $A^D A = P_* = AA^D$ where $P_* = I - P_0$.

With $\text{ind}(A) = 1$ we can verify $A^2 A^D = A$, i.e., the pseudo-inverseness of A^D:

$$A^2 A^D = \begin{bmatrix} 4 & 12 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -3/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A.$$

We also verify $A^D AA^D = A^D$ by direct computation

$$\begin{bmatrix} 1/2 & -3/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & -3/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -3/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or more symbolically by using $AA^D = P_*$ to get

$$A^D AA^D = \begin{bmatrix} 1/2 & -3/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -3/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^D.$$

Example. Compute and compare the Moore-Penrose and Drazin inverses of

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the compact SVD

$$\begin{aligned} A &= U_1 \Sigma_1 V_1^{\mathrm{H}} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{H}} \end{aligned}$$

we compute the Moore-Penrose inverse

$$
\begin{aligned}
A^\dagger &= V_1 \Sigma^{-1} U_1^{\mathrm{H}} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{H}} \\
&= \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.
\end{aligned}
$$

Here the projections

$$
\operatorname{proj}_{\mathscr{R}(A)} = AA^\dagger = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

and

$$
\operatorname{proj}_{\mathscr{R}(A^{\mathrm{H}})} = A^\dagger A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

are not the same because the $\mathscr{R}(A)$ and $\mathscr{R}(A^{\mathrm{H}})$ are not the same (the columns of A and A^{H} span different subspaces).

Using partial fractions we have for the resolvent that

$$
\begin{aligned}
R_A(z) = \frac{1}{z-2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{z-1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
+ \frac{1}{z-0} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \frac{1}{(z-0)^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

The eigenprojections of A are

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, P_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and the eigennilpotent of A is

$$D_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Wedderburn decomposition of A is

$$A = D_0 + AP_* = D_0 + A(I - P_0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 12.9.5, the Drazin Inverse of A is

$$A^D = \frac{1}{2}P_2 + P_1 = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Drazin Inverse is not the same as the Moore-Penrose inverse.

As a verification of A^D we check

$$A^D A = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$AA^D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

that they both agree and are equal $P_* = I - P_0$.

With $\operatorname{ind}(A) = 2$ we verify that $A^3A^D = A^2$ but that $A^2A^D \neq A$:

$$A^3A^D = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A^2,$$

but

$$A^2A^D = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A.$$