Math 521 Lecture #2
§1.1.2: Dimensional Analysis

A basic technique in the initial stage of modeling is dimensional analysis.

This is especially useful when governing principles are not available (as they are in many biological systems today, and as they were in many physical systems years ago).

The basis of dimensional analysis is that the variables in an equation must be dimensional homogeneous, i.e., there can only be certain relationships among the variables.

The cornerstone of dimensional analysis is the Pi Theorem which states that for a physical law giving a relationship among dimensioned quantities there is an equivalent physical law giving a relation among certain dimensionless quantities (often labeled π_1, π_2, etc., and hence the name).

Example 1.1. In an atomic explosion, a large amount of energy is released in a short amount of time in a chunk of space small enough to be considered a point.

From the center a shock wave emerges and spreads outward.

The pressure behind the shock wave is on the order of hundreds of thousands of atmospheres, making the ambient air pressure negligible.

We may assume plausibly that there is a relationship between the radius of the wave front r, the time t, the initial air density ρ and the energy released E.

We are assuming that there is a physical law of the form

$$g(t, r, \rho, E) = 0$$

which postulates a functional relationship among the four dimensioned quantities.

The variable t has the dimension of time, r the dimension of length, ρ the dimensions of mass·length^{-3}, and energy the dimensions of mass·length^2·time^{-2}.

A dimensionless quantity formed from the four dimensioned quantities is

$$\frac{r^5\rho}{t^2E}.$$

By the Pi Theorem, there is an equivalent physical law of the form

$$f\left(\frac{r^5\rho}{t^2E}\right) = 0.$$

This says that the dimensionless quantity is a root of f, that is, that

$$\frac{r^5\rho}{t^2E} = C$$

for some dimensionless constant C.

Solving for r gives

$$r = C\left(\frac{Et^2}{\rho}\right)^{1/5}$$

so that the radius of the wave front r grows as $t^{2/5}$.

Data from explosions confirms this dependence of the radius on the time.

By determining C and ρ, the value of E can them be determined.

Example 1.2. How high can we throw a baseball vertically?

Ignoring air resistance, we may assume that the maximum height h depends on the mass m of the baseball, the acceleration of gravity g, and the velocity v with which we throw the baseball.

We could answer this question using Newtonian mechanics (i.e., an ODE), but we will use dimensional analysis instead.

We are assuming a physical law of the form

$$f(m, g, v, h) = 0$$

relating the four dimensioned quantities which can be written in terms of M (mass), L (length), and T (time).

If Π is a dimensionless quantity that can be formed from the four dimensioned quantities, then

$$\Pi = m^a g^b v^c h^d$$

for real powers a, b, c, d.

In terms of the dimensions of m, g, v, and h, this means that

$$\Pi = M^a(LT^{-2})^b(LT^{-1})^c L^d = M^a L^{a+b+d} T^{-2b-c}.$$

For this to be dimensionless requires that all of the exponents be 0, so that

$$a = 0, \quad b + c + d = 0, \quad -2b - c = 0.$$

[You should recognize that we have obtained a set of linear homogeneous equations, a.k.a. linear algebra.]

Surely $a = 0$, while there are two equations in three variables (hence a free variable).

If we say that c is the free variable, and pick it to be $c = -2$, then $b = 1$ and $d = 1$.

[Recall that every other solution is a scalar multiple of this solution, i.e., there is only one linearly independent solution!]

Thus we have that $\Pi = m^0 g^1 v^{-2} h^1$ is a dimensionless quantity (and this is the only one modulo the exponents being multiplied by the same constant).

An equivalent physical law is then

$$g^1 v^{-2} h^1 = C$$

for some constant C (which can be determined by experiment).

We obtain the relationship

$$h = \frac{Cv^2}{g}$$

which means that maximum height is proportional to the square of the velocity.

If instead, we used Newtonian mechanics, we would have the initial value problem $mu'' = -mg$, $u(0) = 0$, $u'(0) = v$, which solves to give the maximum height of $v^2/2g$, so that $C = 1/2$.

Both methods gave the same conclusion!

Example 1.3. The force F of air resistance on a moving object appears to be related to the speed v of the object and the cross-sectional area A of the object.

The dimensions of force are MLT^{-2}, the dimensions of speed are LT^{-1}, and the dimensions of cross-sectional area are L^2.

The force involves mass while the speed and cross-sectional area do not, so there have to be another variable to consider.

We will add air density ρ whose dimensions are ML^{-3}.

We then assume a relation of the form

$$F = f(\rho, A, v).$$

For this to be dimensional correct, the value of f must have the dimensions of force.

Supposing the $f(\rho, A, v) = k\rho^x A^y v^z$ for a dimensionless quantity k, what powers of ρ, A, and v would combine to give f the dimensions of force?

Symbolically we are dealing with the equation

$$MLT^{-2} = (ML^{-3})^x (L^2)^y (LT^{-1})^z.$$

Equating the exponents of M, L, and T gives a system of non homogeneous linear equations,

$$x = 1, \ -3x + 2y + z = 1, \ -z = -2.$$

There is only one solution to this system, namely $x = 1$, $y = 1$, and $z = 2$, and so

$$F = k\rho A v^2.$$

Thus the force of the air resistance is proportional air density, proportional to the cross-sectional area, while proportional to the square of the velocity.

We will look at the Pi Theorem in a more formal setting next time, followed by more examples.