Math 521 Lecture #3
§1.1.3: The Pi Theorem

Last lecture we illustrated the Pi Theorem through several examples.

Roughly speaking, the Pi Theorem states that a physical law

$$f(q_1, q_2, \ldots, q_m) = 0$$

relating m dimensional quantities $q_1, q_2, \ldots, q_m$ is equivalent to a physical law

$$F(\pi_1, \pi_2, \ldots, \pi_k) = 0$$

relating k dimensionless quantities $\pi_1, \pi_2, \ldots, \pi_k$, where the k dimensionless quantities are formed from the m dimensional quantities.

The benefits of a dimensionless formulation of the physical law are

1. the formula is independent of the units used,
2. there are fewer dimensionless quantities than the dimensioned ones, and
3. important relations among the dimensioned quantities can be discovered.

To give a precise statement of the Pi Theorem, we will need some basic terminology.

The dimensional quantities $q_1, q_2, \ldots, q_m$ are expressed in terms of a minimal set of fundamental dimensions $L_1, L_2, \ldots, L_n$ for $n < m$.

Typical fundamental dimensions are time T, length L, and mass M.

For each quantity q_j, we express its dimensions by

$$[q_j] = L_1^{a_{1j}} L_2^{a_{2j}} \cdots L_n^{a_{nj}}$$

for some choice of exponents $a_{1j}, a_{2j}, \ldots, a_{nj}$.

The quantity q_i is dimensionless if and only if $[q_i] = 1$, i.e., all of the exponents are zero.

For the monomial

$$\pi = q_1^{p_1} q_2^{p_2} \cdots q_m^{p_m},$$

to be dimensionless requires that

$$\begin{aligned} 1 = [\pi] &= [q_1]^{p_1} [q_2]^{p_2} \cdots [q_m]^{p_m} \\ &= (L_1^{a_{11}} L_2^{a_{21}} \cdots L_n^{a_{n1}})^{p_1} (L_1^{a_{12}} L_2^{a_{22}} \cdots L_n^{a_{n2}})^{p_2} \cdots (L_1^{a_{1m}} L_2^{a_{2m}} \cdots L_n^{a_{nm}})^{p_m}. \end{aligned}$$

Thus the power of each L_i must be 0, and we obtain a linear system of homogeneous equations

$$A\vec{p} = 0$$

where $A = (a_{ij})$ is the **dimension matrix**, and $\vec{p} = [p_1, p_2, \ldots, p_m]^T$.

For r the rank of A, the solution set of $A\vec{p} = 0$ (or the kernel of A) is a vector space with dimension $m - r$.

[Recall from linear algebra the result that the rank plus nullity of a matrix equals the number of columns of that matrix.]

This says that the number of linearly independent dimensionless quantities that can be formed from the dimensioned quantities $q_1, q_2, \ldots, q_m$ is $k = m - r$.

For linearly independent $\vec{p}_1, \ldots, \vec{p}_k$ each satisfying $A\vec{p} = 0$, we get k dimensionless quantities $\pi_1, \pi_2, \ldots, \pi_k$.

What assumptions are needed to show the equivalence of the dimensioned physical law $f(q_1, q_2, \ldots, q_m)$ with the dimensionless physical law $F(\pi_1, \pi_2, \ldots, \pi_k) = 0$?

We will assume that $f(q_1, q_2, \ldots, q_m)$ is unit-free in the sense that it is independent of the units chosen for the dimensions.

We are distinguishing between dimension and unit: the dimension of time can have units of seconds, minutes, or hours, etc.; the dimension of mass can have units slugs, grams, kilograms, etc.

The units of a fundamental dimension L_i can be changed by multiplication by the appropriate conversion factor λ_i to obtain $\overline{L_i}$ in a new system of units.

If we write $\overline{L_i} = \lambda_i L_i$ for $i = 1, \ldots, n$ for the change of the system of units, then the units of a dimensioned quantity q are changed in a similar manner.

That is, if $[q] = L_1^{b_1} L_2^{b_2} \cdots L_n^{b_n}$, then $\overline{q} = \lambda_1^{b_1} \lambda_2^{b_2} \cdots \lambda_n^{b_n} q$ gives q in the new systems of units.

Definition 1.4. A physical law $f(q_1, q_2, \ldots, q_m) = 0$ is **unit-free** if for all real positive $\lambda_1, \lambda_2, \ldots, \lambda_n$, we have $f(\overline{q}_1, \overline{q}_2, \ldots, \overline{q}_m) = 0$ if and only if $f(q_1, q_2, \ldots, q_m) = 0$.

Example. The ideal gas law is $pv = nrt$ for p the pressure, v the volume, n the moles, r the gas constant, and t the temperature.

This physical law is $f(p, v, n, r, t) = 0$ for $f(p, v, n, r, t) = pv - nrt$.

We have $[p] = ML^{-1}T^{-2}$, $[v] = L^3$, $[n] = \text{mol}$, $[r] = ML^2T^{-2}\Theta^{-1}\text{mol}^{-1}$, and $[t] = \Theta$.

[Here we use Θ for the dimension of temperature and mol for the dimension of moles.]

Setting $\overline{M} = \lambda_1 M$, $\overline{L} = \lambda_2 L$, $\overline{T} = \lambda_3 T$, $\overline{\text{mol}} = \lambda_4 \text{mol}$, and $\overline{\Theta} = \lambda_5 \Theta$ for real positive λ_i, we have

$$\overline{p} = \lambda_1 \lambda_2^{-1} \lambda_3^{-2},\ \overline{v} = \lambda_2^3 v,\ \overline{n} = \lambda_4 n,\ \overline{r} = \lambda_1 \lambda_2^2 \lambda_3^{-2} \lambda_4^{-1} \lambda_5^{-1} r,\ \overline{t} = \lambda_5 t.$$

This gives

$$\begin{aligned} f(\overline{p}, \overline{v}, \overline{n}, \overline{r}, \overline{t}) &= (\lambda_1 \lambda_2^{-1} \lambda_3^{-2})(\lambda_2^3) pv - (\lambda_4)(\lambda_1 \lambda_2^2 \lambda_3^{-2} \lambda_4^{-1} \lambda_5^{-1})(\lambda_5) nrt \\ &= (\lambda_1 \lambda_2^2 \lambda_3^{-2})(pv - nrt) \\ &= (\lambda_1 \lambda_2^2 \lambda_3^{-2}) f(p, v, n, r, t). \end{aligned}$$

This implies that $f(\overline{p}, \overline{v}, \overline{n}, \overline{r}, \overline{t}) = 0$ if and only if $f(p, v, n, r, t) = 0$, and so the ideal gas law is unit-free.

We can now give the formal statement of the Pi Theorem.

Theorem 1.6. Let $f(q_1, q_2, \ldots, q_m) = 0$ be a unit-free physical law that relates the dimensional quantities $q_1, q_2, \ldots, q_m$. Let $L_1, L_2, \ldots, L_n$ $(n < m)$ be fundamental dimensions with

$$[q_j] = L_1^{a_{1j}} L_2^{a_{2j}} \cdots L_n^{a_{nj}}, \ j = 1, \ldots, m,$$

and let r be the rank of the dimension matrix $A = (a_{ij})$. Then there exists $k = m - r$ (linearly) independent dimensionless quantities $\pi_1, \pi_2, \ldots, \pi_k$ that can be formed from $q_1, q_2, \ldots, q_m$, and the physical law $f(q_1, q_2, \ldots, q_m) = 0$ is equivalent to an equation

$$F(\pi_1, \pi_2, \ldots, \pi_k) = 0$$

expressed solely in terms of the dimensionless quantities.

The Pi Theorem does not validate the unit-free physical law $f(q_1, q_2, \ldots, q_m) = 0$; it only reduces it to a dimensionless form.

Example 1.7. Consider a unit-free physical law $f(t, r, u, e, k, c) = 0$ where

$$[t] = T, \ [r] = L, \ [u] = \Theta, \ [e] = E, \ [k] = L^2T^{-1}, \ [c] = E\Theta^{-1}L^{-3}$$

for the fundamental dimensions of time T, length L, temperature Θ, and energy E.

With the dimensions ordered these ways, the dimension matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The rank of A is 4 so the nullity is 2: two linearly independent vectors in the kernel of A are

$$[-1/2, 1, 0, 0, -1/2, 0]^T, \text{ and } [3/2, 0, 1, -1, 3/2, 1]^T.$$

So two dimensionless quantities are

$$\pi_1 = t^{-1/2} r k^{-1/2} = \frac{r}{\sqrt{kt}} \text{ and } \pi_2 = t^{3/2} u e^{-1} k^{3/2} c = \frac{uc(kt)^{3/2}}{e}.$$

The physical law $f(t, r, u, e, k, c) = 0$ is equivalent to the dimensionless physical law $F(\pi_1, \pi_2) = 0$.

When this can be solved for π_2, then we have $\pi_2 = g(\pi_1)$ for some function g.

We can solve $\pi_2 = uc(kt)^{3/2}e^{-1}$ for u to obtain

$$u = \frac{e}{c(kt)^{3/2}} g\left(\frac{r}{\sqrt{kt}}\right).$$