Math 521 Lecture #4
§1.1.4: Proof of the Pi Theorem

The Pi Theorem is a means by which we reduce a unit-free physical law $f(q_1, q_2, \ldots, q_m) = 0$ to a simpler equivalent physical law $F(\pi_1, \pi_2, \ldots, \pi_k) = 0$.

We have already seen how through finding the kernel of the dimension matrix A we define the dimensionless quantities $\pi_1, \pi_2, \ldots, \pi_k$ for the simpler physical law.

We have not yet seen how we get F from f.

We illustrate the process of getting F from f when there are $m = 4$ dimensioned quantities q_1, q_2, q_3, q_4, there are $n = 2$ fundamental dimensions L_1 and L_2, and the rank of A is $r = 2$.

Consider the unit-free physical law $f(q_1, q_2, q_3, q_4) = 0$.

The entries of the 2×4 dimension matrix $A = (a_{ij})$ are determined by

$$[q_j] = L_1^{a_{1j}} L_2^{a_{2j}},\ j = 1, 2, 3, 4.$$

For a dimensionless quantity

$$\pi = q_1^{p_1} q_2^{p_1} q_3^{p_3} q_4^{p_4},$$

the vector $\vec{p} = (p_1, p_2, p_3, p_4)^T$ satisfies $A\vec{p} = 0$, i.e.,

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} p_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} p_2 + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} p_3 + \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix} p_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the rank of A is 2, we may, by rearranging the indices on the q_j, assume that the first two columns of A are linearly independent.

Then we can write the third and fourth columns of A as linear combinations of the first two columns of A: for constants c_{31}, c_{32}, c_{41}, and c_{42}, we have

$$\begin{aligned} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} &= c_{31} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + c_{32} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}, \\ \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix} &= c_{41} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + c_{42} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}. \end{aligned}$$

The components of $\vec{p}$ then satisfy

$$(p_1 + c_{31}p_3 + c_{41}p_4) \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} + (p_2 + c_{32}p_3 + c_{42}p_4) \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We have here a linear combination of first two columns of A equal the zero vector.

By linear independence of the first two columns of A, the coefficients in the linear combination must both be zero:

$$p_1 + c_{31}p_3 + c_{41}p_4 = 0, \quad p_2 + c_{32}p_3 + c_{42}p_4 = 0.$$

We can solve for p_1 and p_2 in terms of p_3 and p_4, giving

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = p_3 \begin{bmatrix} -c_{31} \\ -c_{32} \\ 1 \\ 0 \end{bmatrix} + p_4 \begin{bmatrix} -c_{41} \\ -c_{42} \\ 0 \\ 1 \end{bmatrix}.$$

The two vectors in this linear combination are a basis for the kernel of A, and give us the two dimensionless quantities

$$\pi_1 = q_1^{-c_{31}} q_2^{-c_{32}} q_3, \quad \pi_2 = q_1^{-c_{41}} q_2^{-c_{42}} q_4.$$

Define a function G by

$$G(q_1, q_2, \pi_1, \pi_2) = f(q_1, q_2, \pi_1 q_1^{c_{31}} q_2^{c_{32}}, \pi_2 q_1^{c_{41}} q_2^{c_{42}}).$$

Observe that $G(q_1, q_2, \pi_1, \pi_2) = 0$ holds if and only if $f(q_1, q_2, q_3, q_4) = 0$, and so we have an equivalent physical law.

Since $f = 0$ is unit-free, it follows that $G = 0$ is also unit-free.

[The dimensionless quantities π_j satisfy $\overline{\pi}_j = \pi_j$ in all systems of units.]

The final step is the show that $G(1, 1, \pi_1, \pi_2) = 0$ is equivalent to $G(q_1, q_2, \pi_1, \pi_2) = 0$, for then we define

$$F(\pi_1, \pi_2) = G(1, 1, \pi_1, \pi_2).$$

Because $G(q_1, q_2, \pi_1, \pi_2) = 0$ is unit-free, we have

$$G(\overline{q}_1, \overline{q}_2, \pi_1, \pi_2) = 0$$

for every choice of the conversion factors λ_1 and λ_2 in

$$\overline{q}_1 = \lambda_1^{a_{11}} \lambda_2^{a_{21}} q_1, \quad \overline{q}_2 = \lambda_1^{a_{21}} \lambda_2^{a_{22}} q_2.$$

Is there a choice of λ_1 and λ_2 so that $\overline{q}_1 = 1$ and $\overline{q}_2 = 1$?

We are solving

$$\lambda_1^{a_{11}} \lambda_2^{a_{12}} q_1 = 1, \quad \lambda_1^{a_{21}} \lambda_2^{a_{22}} q_2 = 1,$$

which is equivalent to solving

$$\begin{aligned} a_{11} \ln \lambda_1 + a_{21} \ln \lambda_2 &= -\ln q_1, \\ a_{21} \ln \lambda_1 + a_{22} \ln \lambda_2 &= -\ln q_2. \end{aligned}$$

This linear system has a unique solution $(\ln \lambda_1, \ln \lambda_2)$ because the coefficient matrix is invertible.

We there is a choice of λ_1 and λ_2.

Thus $F(\pi_1, \pi_2) = 0$ is equivalent to $f(q_1, q_2, q_3, q_4) = 0$.

Example. Recall the ideal gas law $0 = f(p, v, n, r, t) = pv - nrt$ is unit-free.

Here $[p] = ML^{-1}T^{-2}$, $[v] = L^3$, $[n] = \text{mol}$, $[r] = ML^2T^{-2}\Theta^{-1}\text{mol}^{-1}$, and $[t] = \Theta$.

The dimension matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ -1 & 3 & 0 & 2 & 0 \\ -2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

To find the kernel of A we row reduce A to get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So the rank of A is 4, and the nullity of A is 1.

The five components of $\vec{p}$ in $A\vec{p} = 0$ satisfies

$$p_1 = -p_5,\ p_2 = -p_5,\ p_3 = p_5,\ p_4 = p_5,\ p_5 \text{ free }.$$

Thus a basis for the kernel of A is $[-1, -1, 1, 1, 1]^T$ (obtained by taking $p_5 = 1$), and the one dimensionless quantity is

$$\pi_1 = p^{-1}v^{-1}nrt.$$

We form the function G by

$$G(p, v, n, r, \pi_1) = f(p, v, n, r, \pi_1 pvn^{-1}r^{-1}) = pv - nr(\pi_1 pvn^{-1}r^{-1}) = pv(1 - \pi_1).$$

The function F for the simpler equivalent dimensionless ideal gas law is then given by

$$F(\pi_1) = G(1, 1, 1, 1, \pi_1) = 1 - \pi_1.$$

Setting $F(\pi_1) = 0$ implies that $\pi_1 = 1$, so that

$$1 = p^{-1}v^{-1}nrt.$$

But this is nothing more than saying that $pv = nrt$.

Example. Suppose there are at least two (linearly independent) dimensionless quantities $\pi_1, \pi_2, \ldots, \pi_k$ in the dimensionless physical law $F(\pi_1, \pi_2, \ldots, \pi_k) = 0$.

Then we may assume when $\partial F/\partial \pi_k \neq 0$ that π_k is a function of the $\pi_1, \pi_2, \ldots, \pi_{k-1}$ (by the Implicit Function Theorem).

This gives the functional form of the quantity π_k or any of the dimensional quantities that determine π_k.