Math 521 Lecture #6
§1.2.3 The Projectile Problem

Recall that scaling can lead to incorrect approximations, and so care must be taken when neglecting small terms, for they may not be small after all.

Proper scaling is essential in determining orders of magnitude of the terms.

The problem to analyze is the height at time t of a projectile thrust vertically upward from the surface of the earth at time $t = 0$.

Let R be the radius of the earth, M the mass of the earth, m the mass of the projectile, and V the velocity with which the projectile is thrust upward.

Ignoring air resistance, and using Newton's second law and the Newton's universal gravitational law, the height h is governed by

$$m\frac{d^2h}{dt^2} = -G\frac{Mm}{(h+R)^2}, \quad h(0) = 0, \quad h'(0) = V,$$

where G is the universal gravitation constant (roughly 6.6732×10^{-11}).

When $h = 0$, the gravitational force is equal to $-mg$, and therefore

$$\frac{GM}{R^2} = g \text{ or } GM = gR^2.$$

With this substitution and the cancellation of the common m, the ODE becomes

$$\frac{d^2h}{dt^2} = -\frac{R^2 g}{(h+R)^2}.$$

The dimensions of the quantities in this model are

$$[t] = T,\ [V] = LT^{-1},\ [h] = L,\ [g] = LT^{-2},\ [R] = L.$$

There are two fundamental dimensions, time T and length L.

For the above ordering of the dimensioned quantities and the fundamental dimensions, the dimension matrix is

$$\begin{bmatrix} 1 & -1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

With the rank of the dimension matrix being 2, there are 3 dimensionless quantities given by

$$\pi_1 = \frac{h}{R},\ \pi_2 = \frac{t}{R/V},\ \pi_3 = \frac{V}{\sqrt{gR}}.$$

By the Pi Theorem, if there is a physical law relating t, h, R, V, and g (and there is because we the IVP has a unique solution which gives the physical law), then there is an equivalent dimensionless law $G(\pi_1, \pi_2, \pi_3) = 0$.

Supposing that this dimensionless law can be solved for π_1, we obtain

$$\frac{h}{R} = f\left(\frac{t}{R/V}, \frac{V}{\sqrt{gR}}\right).$$

Although it may not seem obvious, we have gained much information through this dimensional analysis.

For instance, to find the time $t_{\max}$ at which the projectile reaches its maximum height, we compute $h'(t)$ and set it to zero and replace t with $t_{\max}$:

$$\frac{\partial f}{\partial \pi_2}\left(\frac{t_{\max}}{R/V}, \frac{V}{\sqrt{gV}}\right) = 0.$$

Supposing that we can solve this for $t_{\max}$ gives

$$t_{\max} = \frac{R}{V}\, F\left(\frac{V}{\sqrt{gR}}\right)$$

for some function F.

So the time to maximum height depends on the dimensionless quantity $V/\sqrt{gR}$.

Now we non-dimensionalize the model through choices of characteristic time and length scales.

For an intrinsic time scale t_c and an intrinsic length scale h_c, define dimensionless variables

$$\bar{t} = \frac{t}{t_c},\ \bar{h} = \frac{h}{h_c}.$$

There are several choices of t_c and h_c determined by the parameters in the model.

For t_c we could take R/V, $\sqrt{R/g}$, or V/g.

For h_c we could take R or V^2/g.

Corresponding to the three scaling choices

$$\begin{aligned}
\bar{t} &= \frac{t}{R/V},\ \bar{h} = \frac{h}{R},\\
\bar{t} &= \frac{t}{\sqrt{R/g}},\ \bar{h} = \frac{h}{R},\\
\bar{t} &= \frac{t}{V/g},\ \bar{h} = \frac{h}{V^2/g},
\end{aligned}$$

are the three dimensionless models

$$\begin{aligned}
\epsilon\frac{d^2\bar{h}}{d\bar{t}^2} &= -\frac{1}{(1+\bar{h})^2},\quad \bar{h}(0) = 0,\quad \frac{d\bar{h}}{d\bar{t}}(0) = 1,\\
\frac{d^2\bar{h}}{d\bar{t}^2} &= -\frac{1}{(1+\bar{h})^2},\quad \bar{h}(0) = 0,\quad \frac{d\bar{h}}{d\bar{t}}(0) = \sqrt{\epsilon},\\
\frac{d^2\bar{h}}{d\bar{t}^2} &= -\frac{1}{(1+\epsilon\bar{h})^2},\quad \bar{h}(0) = 0,\quad \frac{d\bar{h}}{d\bar{t}}(0) = 1,
\end{aligned}$$

where $\epsilon = V^2/gR$ is a dimensionless quantity.

To see the difficulties that arise when an incorrect scaling is used, we suppose that $\epsilon \ll 1$.

By ignoring the term with ϵ, the first dimensionless model becomes

$$(1+\overline{h})^{-2} = 0,\ \overline{h}(0) = 0,\ d\overline{h}/d\bar{t}(0) = 1,$$

which has no solution.

Even though $\epsilon \ll 1$, it might not be that

$$\epsilon \frac{d^2\overline{h}}{d\bar{t}^2} \ll 1$$

because the second derivative might be large, and so we are not justified in ignoring this term in the ODE.

Measuring the height relative to R is not a good choice for small initial velocity.

By ignoring the term with ϵ in the second dimensionless model, we get

$$\frac{d^2\overline{h}}{d\bar{t}^2} = -\frac{1}{(1+\overline{h})^2},\ \overline{h}(0) = 0,\ \frac{d\overline{h}}{d\bar{t}}(0) = 0,$$

which has a concave down, hence negatively valued $\overline{h}$ for $t > 0$ which doesn't fit the projectile being thrust vertically upward.

This illustrates another incorrect use of scaling.

However, if we ignore the term in the third dimensionless model, we get

$$\frac{d^2\overline{h}}{d\bar{t}^2} = -1,\ \overline{h}(0) = 0,\ \frac{d\overline{h}}{d\bar{t}}(0) = 1,$$

which gives the approximate solution of

$$\overline{h} = \bar{t} - \frac{\bar{t}^2}{2}.$$

Undoing the scalings, we get

$$h = -\frac{gt^2}{2} + Vt$$

which is consistent with our experience for a projectile thrust upward with small initial velocity.

Measuring height relative to V^2/g is a good selection of the length scale because the maximum height is about $V^2/2g$.

Measuring time relative to V/g is a good selection for the time scale because the time to achieve the maximum height is roughly V/g.