Math 521 Lecture #7
§1.3.1: Review of Elementary Methods

Recall that an initial value problem (or IVP for short),

$$u' = f(t,u),\ u(t_0) = u_0,$$

has a unique solution when f and $\partial f/\partial u$ are continuous on an open rectangle containing the point (t_0, u_0).

In some cases, the ODE $u' = f(t,u)$ can be solved, at least implicitly, with an arbitrary constant whose value is chosen to match the initial condition.

The method used to solve and first-order ODE depends on its type.

Separable First-Order Equations. An ODE $u' = f(t,u)$ is separable when

$$f(t,u) = g(t)h(u).$$

For then we can separate the variables in the ODE to get

$$\frac{du}{h(u)} = g(t)dt.$$

Integration of this gives implicitly defined solutions

$$\int \frac{du}{h(u)} = \int g(t)dt + C.$$

Example. The logistic equation $u' = u(1-u)$ is separable, giving implicitly defined solutions by

$$\int \frac{du}{u(1-u)} = t + C.$$

By partial fractions,

$$\frac{1}{u(1-u)} = \frac{1}{u} - \frac{1}{1-u},$$

and so

$$\int \frac{du}{u(1-u)} = \ln|u| + \ln|1-u| = \ln\left|\frac{u}{1-u}\right|.$$

The implicitly defined solutions of the ODE are

$$\ln\left|\frac{u}{1-u}\right| = t + C \ \Rightarrow\ \frac{u}{1-u} = (\pm e^C)e^t.$$

Can we get explicitly defined solutions from this?

First-Order Linear Equations. A first-order ODE $u' = f(t,y)$ if is can be written in the form

$$u' + p(t)u = g(t).$$

Multiplying a linear ODE through by the integration factor

$$\exp\left\{\int p(t)dt\right\}$$

turns the left-hand side into a total derivative (via the product rule),

$$\frac{d}{dt}\left(u\exp\left\{\int p(t)dt\right\}\right) = q(t)\exp\left\{\int p(t)dt\right\}.$$

Integration and solving for u explicitly gives

$$u = \exp\left\{-\int p(t)dt\right\}\left(\int q(t)\exp\left\{\int p(t)dt\right\} + C\right).$$

Example. The ODE $u' - 2tu = 1$ is linear, and so its general solution is

$$u = \exp\left\{\int 2tdt\right\}\left(\int \exp\left\{-2tdt\right\} + C\right) = e^{t^2}\left(\int e^{-t^2}dt + C\right).$$

If we were to impose an initial condition $u(a) = b$, then the solution of the IVP is

$$u = \exp\left(t^2 - a^2\right)\left(\int_a^t e^{s^2}ds + b\right).$$

Second-Order Linear Equations. A second-order equation $F(t, u, u', u'') = 0$ is linear if it can be written in the form

$$u'' + p(t)u' + q(t)u = f(t).$$

Solutions of the homogeneous equation (i.e., $f(t) = 0$) form a two dimensional vector space.

Two linearly independent solutions of the homogeneous equation (i.e., $f(t) = 0$) form a basis of this two dimensional vector space, and are called a fundamental set of solutions.

Recall that two homogeneous solutions are linearly independent if and only if they have a nonzero Wronskian

$$W(t) = u_1(t)u_2'(t) - u_1'(t)u_2(t).$$

Once a fundamental set of homogeneous solutions u_1, u_2 is obtained, a particular solution of the non-homogeneous equation is given by the variation of parameters formula,

$$u_P(t) = -u_1(t)\int \frac{u_2(t)f(t)}{W(t)}dt + u_2(t)\int \frac{u_1(t)f(t)}{W(t)}dt.$$

The general solution of the non-homogeneous equation is then

$$u = c_1u_1(t) + c_2u_2(t) + u_P(t).$$

When the functions $p(t)$ and $q(t)$ are real analytic, we can find two linearly independent power series solution of the homogeneous equation.

In simplest case, when we have $au''+bu'+cu = 0$, the guess $u = e^{rt}$ gives the characteristic equation

$$ar^2 + br + c = 0,$$

whose roots determine the form of the fundamental set.

When the roots r_1 and r_2 are real and distinct, two solutions are $u_1 = e^{r_1 t}$ and $u_2 = e^{r_2 t}$.

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When the roots $\lambda \pm i\mu$ are complex conjugate, two real valued solutions are $u_1 = e^{\lambda t}\cos(\mu t)$ and $u_2 = e^{\lambda t}\sin(\mu t)$.

Another case when a fundamental set of solutions can be obtained is when we have $t^2u'' + \alpha tu' + \beta u = 0$, known as the Cauchy-Euler equation.

The guess $u = t^r$ results in the indicial equation $r^2 + (\alpha - 1)r + \beta = 0$.

It is left to you to formulate the fundamental set of solutions in the three cases of the roots.

Nonlinear Second-Order Equations. Some nonlinear second-order equations can be solved by reduction to a first-order equation.

Consider the equation $mx'' = F(t, x, x')$ for position x and mass m.

If F does not depend on x, then the substitution $y = x'$ reduces the second-order equation to the first-order equation $y' = F(t, y)$.

Once this is solved for y, then solving the separable first-order equation $y = x'$ gives x.

If, instead, F does not depend on t, then setting $y = x'$ and using the chain rule to get

$$x'' = \frac{d}{dt}\frac{dx'}{dt} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y(dy/dx),$$

gives

$$my\frac{dy}{dx} = F(x, y).$$

Once this is solved for y as a function of x, then solving the separable first-order equation $y = x'$ gives x.

Another case is when F depends only on x, and hence is conservative, i.e., there is a potential function $V(x)$ such that

$$F(x) = -\frac{dV}{dx}.$$

The substitution $y = x'$, $x'' = y(dy/dx)$ then gives the separable first-order equation

$$my\frac{dy}{dx} = F(x).$$

Integration immediately gives

$$\frac{my^2}{2} + V(x) = E$$

where the constant of integration E is the total energy in the system (kinetic energy plus potential energy).