Math 521 Lecture #8
§1.3.2: Stability and Bifurcation, Part I

The first-order autonomous equation $u' = f(u)$ appears in many applications.

Because it is separable, we obtain

$$\int \frac{du}{f(u)} = t + C.$$

In some cases we can find the antiderivative of $1/f(u)$ to get to implicitly defined analytic solutions.

For instance, if $f(u) = u^2(1-u)$, then through partial fractions,

$$\frac{1}{f(u)} = \frac{1}{u^2(1-u)} = \frac{u+1}{u^2} + \frac{1}{1-u} = \frac{1}{u} + \frac{1}{u^2} + \frac{1}{1-u},$$

so that solving $u' = f(u)$ gives implicitly defined analytic solutions

$$\ln|u| - \frac{1}{u} - \ln|1-u| = t + C.$$

What do the graphs of these solutions look like? We can not tell easily from this analytic expression since there is no way to solve explicitly for u.

There is a better, more geometric approach to get the qualitative information.

You might recall (from Math 334 or its equivalent) that this geometric approach starts by first identifying the equilibrium solutions of $u' = f(u)$, which are the constants functions $u = u^*$ for which $f(u^*) = 0$.

We will always assume that equilibria are isolated, i.e., if u^* is an equilibrium solution, then it is the only one in an open interval containing u^*.

On a phase line (the u-axis) we place dots at the locations of the equilibria.

Next in the geometric approach, we find the sign of $f(u)$ between consecutive equilibria, for the sign of f tells us the value of u', and hence if the solutions are increasing or decreasing.

We draw arrows on the phase line indicating whether the solutions between two consecutive equilibria are increasing or decreasing.

The phase line gives information about the nature of each equilibria, whether it is locally asymptotically stable, or if it is unstable.

To be precise, and isolated equilibrium u^* of $u' = f(u)$ is locally asymptotically stable (or an attractor) if there exists an open interval I containing u^* such that each solution $u(t)$ of $u' = f(u)$ with $u(0) \in I$ satisfies

$$\lim_{t\to\infty} u(t) = u^*.$$

There is an easy test by which we can determine whether an isolated equilibrium is locally asymptotically stable or not: If $f'(u^*) < 0$, then the sign of $f(u)$ switches from positive for u smaller than u^* to negative for u bigger than u^*, and so the equilibrium u^* is locally asymptotically stable.

What can we conclude if $f'(u^*) > 0$? If $f'(u^*) = 0$? [For the first, the equilibrium is an unstable repeller – all nearby solutions move away from u^*. For the second, we can not conclude anything, but need more information.]

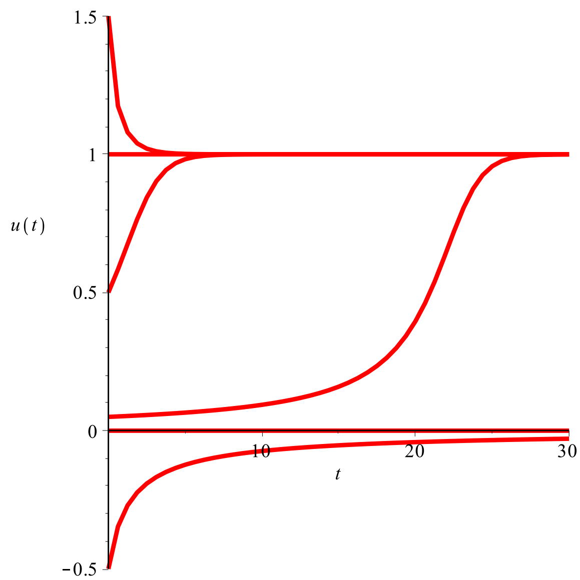
Example. The equation $u' = u^2(1-u)$ has equilibria at $u = 0$ and $u = 1$.

With $f(u) = u^2(1-u) = u^2 - u^3$, we have $f'(u) = 2u - 3u^2 = u(2-3u)$, so that $f'(0) = 0$ and $f'(1) = -1$.

Thus $u = 1$ is an attractor, while we need more information to decide what stability $u = 0$ has.

The sign of f for $u < 0$ is positive and the sign of f for $0 < u < 1$ is positive, so the equilibrium $u = 0$ is semi-stable (or what the book calls hyperbolic).

Below is a time-series for this equation, i.e., plots of the graphs of solutions (generated numerically).



Notice that a solution starting with $u(0)$ small and positive, stays small for some time before rapidly increasing towards 1.

FYI: The equation $u' = u^2(1-u)$ is a model for the combustion of a chemical reactant (Reiss (1980), Kassoy, Kapila).

Example 1.24. A model for a population $p(t)$ with logistic growth, subject to harvesting at a constant rate H is

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right) - H$$

where r is the intrinsic growth rate and K is the carrying capacity for the population.

For fixed r and k, how do the solutions change as H is varied?

With three parameters r, K, and H, we non-dimensionalize the equation by introducing a scaled population u and a scaled time τ by

$$u = \frac{P}{K}, \ \tau = \frac{t}{1/r} = rt.$$

The equation becomes

$$u' = u(1-u) - h$$

where

$$h = \frac{H}{rK}$$

is a dimensionless harvesting constant.

Applying the geometric analysis approach, we identify the equilibria (if any):

$$0 = u(1-u) - h \ \Rightarrow \ u^* = \frac{1 \pm \sqrt{1-4h}}{2}.$$

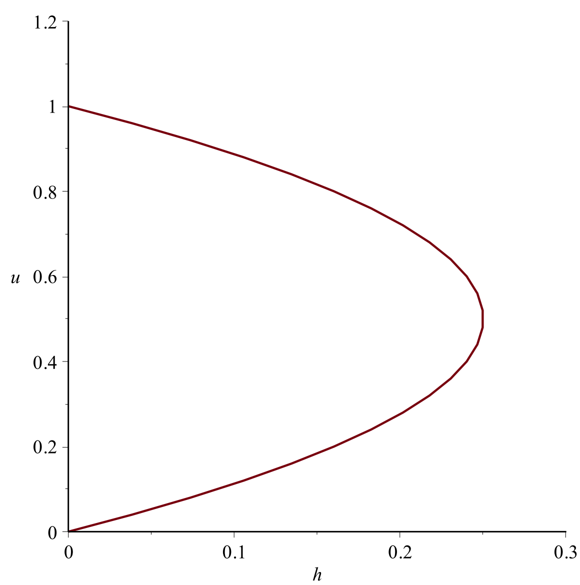
If $h > 1/4$, there are no equilibria, and $u' < 0$ so that the population decays to extinction.

If $0 < h < 1/4$, there are two positive equilibria, each of which represents a balance between growth and harvesting.

By determining the sign of $u(1-u) - h$ away from the two equilibria, we conclude that the larger equilibrium is locally asymptotically stable, while the smaller equilibrium is unstable.

If the population is below the unstable equilibrium, the population decays to extinction, but if the population is above the unstable equilibrium, the population tends toward the attractor equilibrium.

Plotting the ordered pairs (h, u^*) we obtain the following graph.



This graph is called a bifurcation diagram where h is a bifurcation parameter.

As h increases from 0 to 1/4, the two branches of equilibria (the upper branch with attractors and the lower branch with repellers), coalesce into one semistable equilibrium at $h = 1/4$.

As h increases past 1/4, the equilibrium disappears.

We say a bifurcation occurs at $h = 1/4$: the phase lines for all $0 < h < 1/4$ are similar, while for $h > 1/4$, the phase lines are all similar, but different from those for $0 < h < 1/4$.

The phase line bifurcates, or changes, at $h = 1/4$.