Math 521 Lecture #9
§1.3.2: Stability and Bifurcation, Part II

Example 1.25. We model a plant-herbivore system and determine its bifurcation diagram.

We assume that the plant biomass $P(t)$ grows logistically with growth rate r and carrying capacity K.

We assume there are a fixed number H of herbivores who consume plant biomass at the rate of

$$\frac{aP}{1+bP}$$

per herbivore, where a has dimensions of per time per herbivore and b has dimensions of per plant biomass.

The consumption rate of the herbivores limits to a/b as $P \to \infty$.

Under the assumptions, a model for the plant-herbivore system is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \frac{aP}{1+bP}H.$$

There are five parameters in this model, namely r, K, a, b, and H.

We non-dimensionalize the model through the dimensionless quantities

$$\tau = rt,\ N = \frac{P}{K}.$$

The model then becomes

$$\frac{dN}{d\tau} = N(1-N) - \frac{hN}{1+cN}$$

for the two dimensionless parameters

$$h = \frac{aH}{r},\ c = Kb.$$

It is reasonable to assume that the plant biomass has a large carrying capacity, so that we can safely assume $c > 1$ and fixed.

Our interest is in what happens as the number of herbivores varies, so h is the bifurcation parameter.

An equilibrium N^* of the model is a solution of

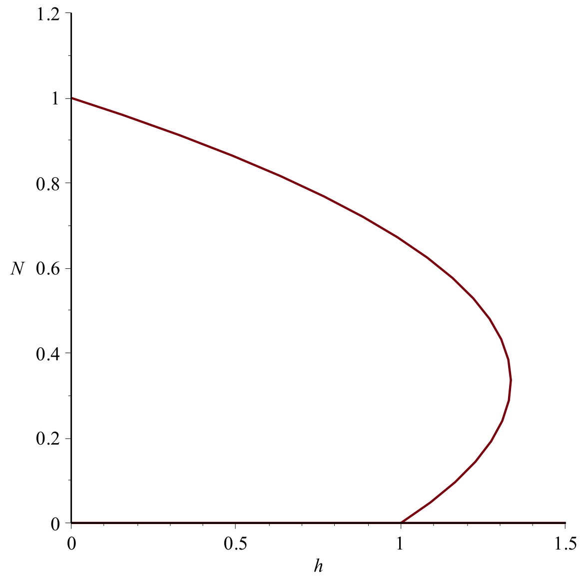
$$N^*\left[(1-N^*) - \frac{h}{1+cN^*}\right] = 0.$$

This gives $N^* = 0$ (for all values of h) and the solutions (if any) of

$$(1-N^*)(1+cN^*) - h = 0.$$

We could solve this for N^* explicitly (via the quadratic formula) as functions of h and plot these curves.

Instead we plot the curve $h = (1 - N^*)(1 + cN^*)$, which gives the same thing, along with $N^* = 0$. [The nontrivial curve is for $c = 3$.]



Notice that $h = 0$ when $N^* = 1$ and $h = 1$ when $N^* = 0$ (regardless of the value of c).

The largest value of h for which there is an equilibrium can be found by setting the derivative of $h = (1 - N^*)(1 + cN^*)$ (with respect to N^*) to zero:

$$h = \frac{(c+1)^2}{4c} \text{ at } N^* = \frac{c-1}{2c}.$$

Through geometric analysis, we learn that each equilibrium on the 0 branch on the bifurcation diagram is unstable for $0 < h < 1$ and locally asymptotically stable for $h > 1$.

Each equilibrium on the upper branch (starting at $(0, 1)$ and down to the "top" of the parabola) is locally asymptotically stable, while on the lower branch (from the "top" of the parabola down to the point $(1, 0)$) is unstable.

The bifurcation diagram includes the words "stable" on the locally asymptotically stable branches, and "unstable" on the unstable branches.

For $h > (c+1)^2/4c$, the locally asymptotically stable equilibrium at $N^* = 0$ is globally asymptotically stable, i.e., every solution has the property that $N \to 0$ as $t \to \infty$.

Example 1.27. We model a non-isothermal tank reactor for determining the temperature $\overline{\theta}$ and the concentration $\overline{c}$ of a heat-releasing, chemically reacting substance.

We assume that the tank is continuously stirred to maintain a uniform temperature and uniform chemical concentration.

The tank, of fixed volume V, is continuously fed a chemical reactant at concentration c_i and temperature θ_i in a stream with constant flow rate q.

After mixing and reacting, the products are removed at the same flow rate q.

We assume the exothermal reaction is first-order and irreversible, and that the reactant disappears at the rate of

$$-k\bar{c}e^{-A/\bar{\theta}}$$

for constants A (with dimension temperature) and k (with dimension per time).

We assume the amount of heat released is given by

$$hk\bar{c}e^{-A/\bar{\theta}}$$

where h is a positive constant (the specific heat of reaction) measured in energy per mass.

We have two quantities to model, the concentration $\bar{c}(\bar{t})$ and the temperature $\bar{\theta}(\bar{t})$.

For the concentration, we use mass balance to obtain

$$V\frac{d\bar{c}}{d\bar{t}} = qc_i - q\bar{c} - Vk\bar{c}e^{-A/\bar{\theta}}.$$

For the temperature, we use heat balance to obtain

$$VC\frac{d\bar{\theta}}{d\bar{t}} = qC\theta_i - qC\bar{\theta} + hVk\bar{c}e^{-A/\bar{\theta}}$$

where C is the heat capacity of the mixture, measured in energy per volume per temperature.

To non-dimensionalize these equations, we introduce the dimensionless variables

$$t = \frac{\bar{t}}{V/q},\ \theta = \frac{\bar{\theta}}{\theta_i},\ c = \frac{\bar{c}}{c_i},$$

and the dimensionless parameters

$$\mu = \frac{q}{kV},\ b = \frac{hc_i}{C\theta_i},\ \gamma = \frac{A}{\theta_i}.$$

The system of two first-oder equations becomes

$$\frac{dc}{dt} = 1 - c - \frac{ce^{-\gamma/\theta}}{\mu},$$
$$\frac{d\theta}{dt} = 1 - \theta + \frac{bce^{-\gamma/\theta}}{\mu}.$$

At first glance, it looks impossible to solve this nonlinear systems of ODEs.

But if we multiply the first one through by the constant b, and add this to the second equation we get

$$\frac{d(\theta + bc)}{dt} = 1 + b + (\theta + bc).$$

This integrates to

$$\theta + bc = 1 + b + De^{-t}$$

for a constant D.

Assuming that $\theta + bc$ at $t = 0$ is $1 + b$, the constant D takes the value 0, so that $\theta + bc = 1 + b$. [This choice of initial condition is to insure the resulting equation is autonomous.]

Thus $bc = 1 + b - \theta$, so the the heat balance equation becomes

$$\frac{d\theta}{dt} = 1 - \theta - \frac{(1 + b - \theta)e^{-\gamma/\theta}}{\mu}.$$

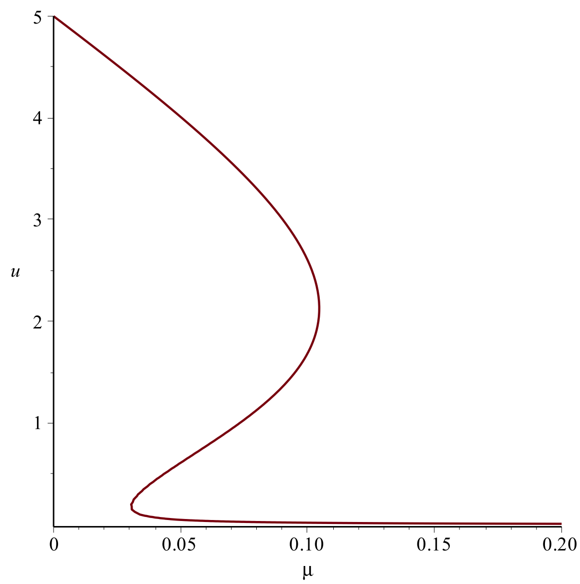
Changing the dependent variable by $u = \theta - 1$ gives

$$\frac{du}{dt} = -u + \frac{(b-u)e^{-\gamma/(u+1)}}{\mu} = \frac{-\mu u + (b-u)e^{\gamma/(u+1)}}{\mu}.$$

The parameter μ is the ratio of the flow rate q and the reaction rate kV.

Fixing the parameters b and γ, we use μ as a bifurcation parameter.

Solving the equation for its equilibrium solutions is not possible in an analytic manner, so instead we use numerical methods to plot the curve in the bifurcation diagram. [The curve is for $b = 5$ and $\gamma = 8$.]



A geometric analysis reveals that the equilibrium is locally (globally if μ small enough) asymptotically stable on the branch connecting $(0, 5)$ to the first vertical tangent, unstable from the first vertical tangent to the second vertical tangent, and locally (globally if μ large enough) asymptotically stable on the remainder part of the curve.