Math 521 Lecture #10
§2.1: Phase Plane Phenomena

Most problems require more than one state variable to describe them.

We extend the geometric analysis of first-order autonomous equations $u' = f(u)$ to a system of two first-order autonomous equations

$$x' = P(x, y),\ y' = Q(x, y)$$

in two state variables x and y.

To guarantee that the system has a unique solution satisfying initial conditions $x(t_0) = \xi_0$ and $y(t_0) = \eta_0$ on some open interval containing t_0, we assume that P and Q are C^1 functions in a neighborhood of (ξ_0, η_0), i.e., their partial derivatives are continuous on an open set containing (ξ_0, η_0).

There are two ways to graphically present the solutions $x(t)$ and $y(t)$ of the system.

The first is as **time series**, which is the graph of x versus t and the graph of y versus t on the same axes.

The second is as a **parametric curve** $(x(t), y(t))$ in the xy-plane, which is called an orbit, path, or trajectory of the system.

It is with this second presentation that we apply our geometric analysis to the system.

An equilibrium (x_0, y_0) is a constant (or steady state) solution, or critical point of the system, that is found by solving simultaneously

$$P(x, y) = 0,\ Q(x, y) = 0.$$

No orbit can pass through an equilibrium at some finite time, for this would violate the uniqueness of solutions of initial value problems.

So any orbit that approaches an equilibrium must do so in infinite time.

Non-equilibrium orbits are to a large extend determined by the orbits near the equilibria.

The **phase plane diagram** or **phase portrait** is obtained by sketching enough sample orbits to give the qualitative nature of all orbits.

Example (2.4,2.5). According to Newtonian mechanics, the equation governing a particle of mass m moving in one dimension x under a force $f(x, x')$ is

$$mx' = f(x, x').$$

This second-order equation can be converted into a system of two first-order equations by setting $y = x'$, for then

$$x' = y,\ y' = x'' = \frac{f(x, x')}{m}.$$

Recall that if f only depends on x, then the force is conservative and there is a potential function $V(x)$ such that $V'(x) = -f(x)$.

In this case, we have the total energy

$$E = \frac{my^2}{2} + V(x)$$

that is conserved along each orbit $(x(t), y(t))$ of the system,

$$\frac{dE}{dt} = my(t)y'(t) + V'(x(t))x'(t) = y(t)f(x(t)) - y(t)f(x(t)) = 0.$$

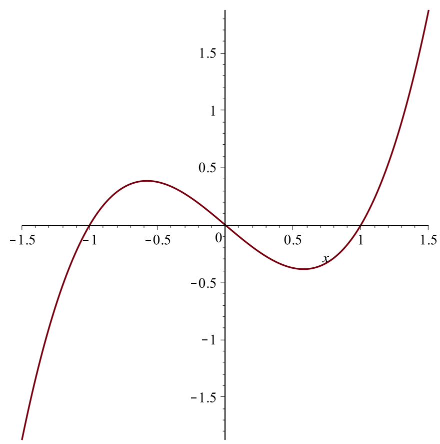
This says that the level curves the total energy are orbits of the system, and using these level curves we can sketch the phase plane diagram.

The level curve corresponding to a fixed value of E is given explicitly by

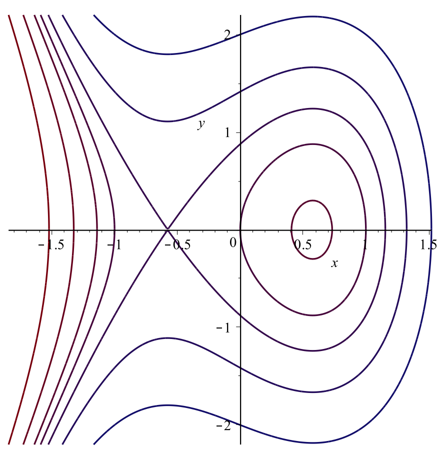
$$y = \pm\sqrt{\frac{2}{m}\big(E - V(x)\big)}.$$

Only for those values of x with $E - V(x) \geq 0$ can an orbit $(x(t), y(t))$ exist at energy E.

For instance, the graph of $V(x) = x^3 - x$ is



and the level curves for strategically chosen values of E are



The equilibria are of the form $(x^*, 0)$ where x^* is a critical point of V.

Surrounding the equilibrium at $(x^*, 0)$ for $x^* > 0$ are **periodic orbits**.

The energy of these periodic orbits, or oscillatory solutions, belongs to the open interval (E_*, E^*) where E_* is the local minimum of V and E^* its local maximum.

The equilibrium at $(0, x^*)$ for $x^* > 0$ appears to be "stable" while the equilibrium at $(0, x^*)$ for $x^* < 0$ appears to be "unstable."

Definition 2.6. Suppose that (a, b) is an isolated equilibrium of $x' = f(x, y)$ and $y' = g(x, y)$.

(i) The equilibrium (a, b) is (Lyapunov) **stable** if for each $\epsilon > 0$ there exists $\delta > 0$ such that for every orbit starting at $t = t_0$ within δ of (a, b) remains within ϵ of (a, b) for all $t \geq t_0$.

(ii) The equilibrium (a, b) is **locally asymptotically stable** if it is stable and there exist $\nu > 0$ such for every orbit starting at $t = t_0$ within ν of (a, b) approaches (a, b) as $t \to \infty$.

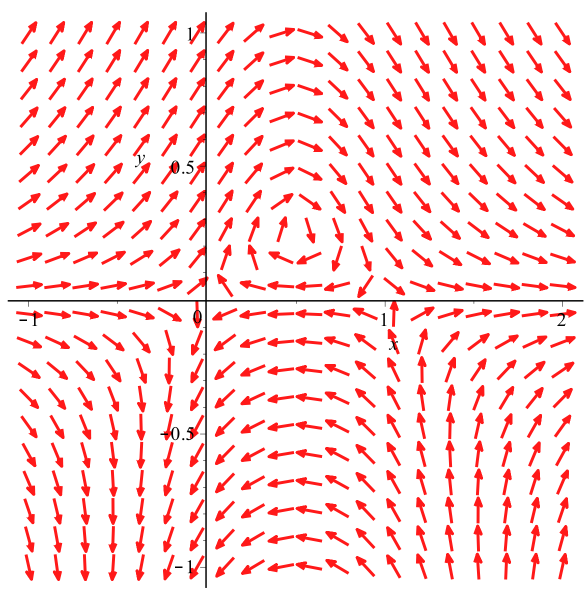
(iii) The equilibrium (a, b) is **unstable** if it is not stable.

We now describe some other techniques for plotting the phase portrait of a system.

Example. Sketch the phase portrait for

$$x' = -x + y + x^2, \; y' = y - 2xy.$$

The system defines a vector field on the plane: at each point (x, y) we plot a small vector in the direction of $(-x + y + x^2, y - 2xy)$.



We first locate the equilibria by setting

$$-x + y + x^2 = 0, \; y - 2xy = 0.$$

The second of these implies that $y = 0$ or $x = 1/2$.

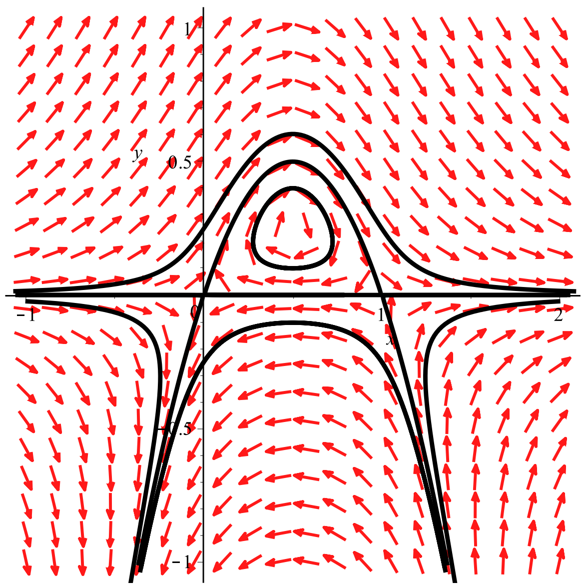
With $y = 0$ in the first, we get $-x + x^2 = 0$, so that $x = 0, 1$, and with $x = 1/2$, we get in the first that $y = 1/4$.

So there are three equilibria: $(0, 0)$, $(1, 0)$, and $(1/2, 1/4)$.

These equilibria are where the **nullclines** $-x+y+x^2 = 0$ (where $x' = 0$) and $y-2xy = 0$ (where $y' = 0$) intersect.

These nullclines divide the phase plane into regions where x' and y' have the same signs (both positive, one positive and one negative, or both negative).

With this information we can sketch the phase portrait.



It appears that the equilibrium at $(1/2, 1/4)$ is stable while those at $(0, 0)$ and $(0, 1)$ are unstable.

As with the mechanical system with a potential energy, we might be able to get an implicit description of the orbits by writing the system as a first-order ODE of the form

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q(x,y)}{P(x,y)} = \frac{y - 2xy}{-x + y + x^2}.$$

We may solve this equation implicitly by $H(x, y) = c$ if there exist $H(x, y)$ such that

$$\frac{\partial H}{\partial x} = y - 2xy, \ \frac{\partial H}{\partial y} = -(-x + y + x^2).$$

Such an H exists (remember exact first-order equations?) and is given by

$$H(x, y) = x^2 y - xy + \frac{y^2}{2}.$$

The level curves $H(x, y) = c$ give the orbits of the system.

We can linearize the system at an equilibrium, say at $(1/2, 1/4)$.

To do this, we write $x(t) = 1/2 + u(t)$ and $y(t) = 1/4 + v(t)$ where $u(t)$ and $v(t)$ represent small perturbations from the equilibrium.

Substitution of these perturbations into the system gives

$$\begin{aligned}
u' &= P(1/2+u, 1/4+v) = -(1/2+u) + (1/4+v) + (1/2+u)^2, \\
&= v + u^2, \\
v' &= Q(1/2+u, 1/4+v) = (1/4+v) - 2(1/2+u)(1/4+v) \\
&= -(1/2)u + 2uv.
\end{aligned}$$

By Taylor's Theorem

$$\begin{aligned}
P(1/2+u, 1/4+v) &= P(1/2, 1/4) + P_x(1/2, 1/4)u + P_y(1/2, 1/4)v + \cdots, \\
Q(1/2+u, 1/4+v) &= Q(1/2, 1/4) + Q_x(1/2, 1/4)u + Q_y(1/2, 1/4)v + \cdots.
\end{aligned}$$

Here $P_x = -1 + 2x$, $P_y = 1$, $Q_x = -2y$, and $Q_y = 1 - 2x$, so that the **linearization** of the system at $x = 1/2$, $y = 1/4$ is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The eigenvalues of the matrix are pure imaginary, and so the equilibrium $(1/2, 1.4)$ is a linear center.