Math 521 Lecture #11
§2.2: Linear Systems

We review the theory of linear systems of first-order homogeneous equations from Math 334 (or its equivalent).

For $\vec{x} = (x_1, x_2)^T$, consider

$$\vec{x}' = A\vec{x}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det A \neq 0.$$

Substitution of the guess $\vec{x}(t) = \vec{v}e^{\lambda t}$ leads the spectral problem

$$A\vec{v} = \lambda\vec{v}.$$

An eigenvalue λ is a root of the characteristic equation

$$\det(A - \lambda I) = 0,$$

and a corresponding eigenvector is a non-zero vectors in $\ker(A - \lambda I)$.

The characteristic equation is

$$\lambda^2 - (\text{tr } A)\lambda + \det A = 0.$$

The assumption $\text{set}(A) \neq 0$ implies that 0 is not an eigenvalue of A (for recall that the product of the eigenvalues of A is the determinant of A).

There are several cases for the form of the general solution of $\vec{x}' = A\vec{x}$.

Case I. A has two distinct real eigenvalues λ_1 and λ_2.

Then there are two linearly independent eigenvectors $\vec{v}_1$ and $\vec{v}_2$ (corresponding to λ_1 and λ_2 respectively).

The general solution is $\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$ for arbitrary constants c_1 and c_2.

The type and stability of the equilibrium at the origin is determined by the signs of λ_1 and λ_2; the origin is an asymptotically stable node if $\lambda_1 < 0$ and $\lambda_2 < 0$; the origin is a saddle point if $\lambda_1\lambda_2 < 0$ (opposite signs); and the origin is an unstable node if $\lambda_1 > 0$ and $\lambda_2 > 0$.

Case II. A has a real repeated eigenvalue λ.

If there are two linearly independent eigenvectors $\vec{v}_1$ and $\vec{v}_2$ for λ, then the general solution is

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda t} + c_2\vec{v}_2 e^{\lambda t},$$

and the equilibrium at the origin is a proper node, asymptotically stable if $\lambda < 0$ and unstable if $\lambda > 0$.

If there is only one linearly independent eigenvector $\vec{v}$ (i.e., the dimension of $\text{mer}(A - \lambda I) = 1$), then the general solution has the form

$$\vec{x}(t) = c_1\vec{v}e^{\lambda t} + c_2(\vec{w} + t\vec{v})e^{\lambda t}$$

where $\vec{w}$ is a generalized eigenvector, i.e., a solution of

$$(A - \lambda I)\vec{w} = \vec{v}.$$

The equilibrium at the origin is an improper node, asymptotically stable if $\lambda < 0$, and unstable if $\lambda > 0$.

<u>Case III</u>. A has complex conjugate eigenvalues $\lambda = \alpha \pm \beta i$ for $\beta \neq 0$.

Corresponding complex eigenvectors are of the form $\vec{w} \pm i\vec{v}$, and one complex solution is

$$(\vec{w} + i\vec{v}) \exp\big((\alpha + i\beta)t\big).$$

Using Euler's Formula $\exp(i\theta) = \cos\theta + i \sin\theta$ we can recover two linearly independent real-valued vector solutions from the one complex-valued vector solution, and hence the general solution

$$\vec{x}(t) = c_1 e^{\alpha t}(\vec{w}\cos\mu t - \vec{v}\sin\mu t) + c_2 e^{\alpha t}(\vec{w}\sin\mu t + \vec{v}\cos\mu t).$$

The equilibrium at the origin is an asymptotically stable spiral point when $\alpha < 0$, a stable center when $\alpha = 0$, and an unstable spiral point when $\alpha > 0$.

We can now characterize when the equilibrium at the origin of a linear system is asymptotically stable.

Theorem 2.7. The equilibrium at the origin of $\vec{x}' = A\vec{x}$ (with $\text{set}(A) \neq 0$) is asymptotically stable if and only if every eigenvalue of A has negative real part.

We can determine when the eigenvalues of A has negative real part in terms of $p = \text{tr } A$ and $q = \det A$.

From $\det(A - \lambda I) = \lambda^2 - (\text{tr } A)\lambda + \det A$, the roots of the characteristic equation are

$$\lambda = \frac{p \pm \sqrt{p^2 - 4q}}{2}.$$

The eigenvalues of A have negative real parts if and only if $p < 0$ and $q > 0$, or in other words, if and only if $\text{tr } A < 0$ and $\det A > 0$.

Example. Find the general solution of

$$\vec{x}\,' = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

The characteristic equation for A is $\lambda^2 - 2\lambda - 3 = 0$, whose roots are $\lambda_1 = -1$ and $\lambda_2 = 3$.

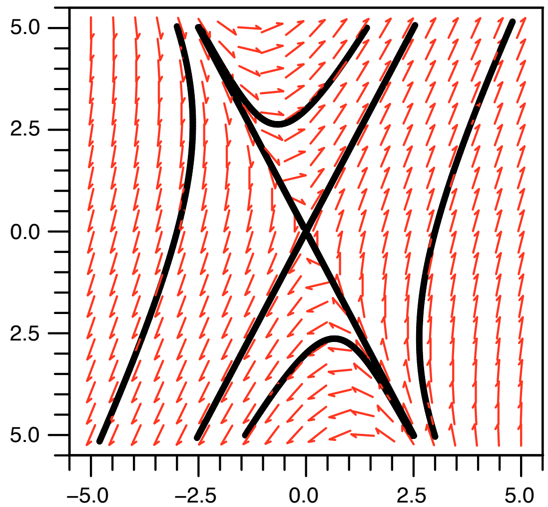
Corresponding eigenvectors for these real distinct eigenvalues are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The general solution is

$$\vec{x}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)} = c_1 \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

We can sketch the phase portrait over the vector field for this system.



The two linearly independent vector solutions we found form the straight line solutions that make the X through the origin.

The equilibrium at the origin is a saddle point.

Example. At what value(s) of the parameter a does the stability of the equilibrium at the origin change for

$$\vec{x}\,' = \begin{bmatrix} a & 1 \\ 2 & -1 \end{bmatrix}.$$

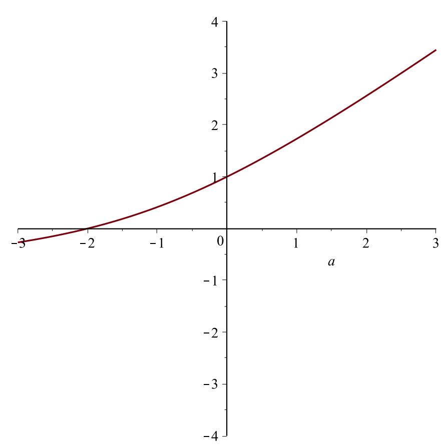
The characteristic equation of A is

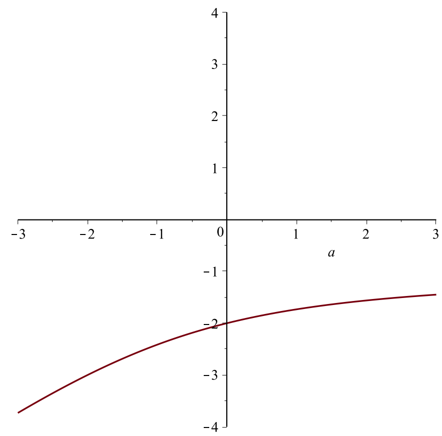
$$\lambda^2 - (a-1)\lambda - (a+2) = 0,$$

and so the eigenvalues of A are

$$\lambda = \frac{a - 1 \pm \sqrt{(a-1)^2 + 4(a+2)}}{2}.$$

One eigenvalue of A is always negative, while the other switched sign at $a = 2$. Here are the graphs of the eigenvalues as functions of a.





Note that the function under the root sign, $(a-1)^2 + 4(a+2) = a^2 + 2a + 9$ is always positive: its critical point is $a = -1$ at which is has a global minimum of 8.

For $a < -2$, the equilibrium at the origin is an asymptotically stable node (tr $A = a-1 < 0$ and det $A = -a-2 > 0$).

For $a = -2$, one eigenvalue is 0, so the equilibrium at the origin is degenerate (i.e., det $A = 0$).

For $a > -2$, the eigenvalues are real and of opposite sign, so the equilibrium at the origin is a saddle point.