Math 521 Lecture #12
§2.3: Nonlinear Systems

In analyzing a nonlinear system

$$x' = P(x, y), \quad y' = Q(x, y),$$

to what extend does its linearization

$$\vec{x}\,' = A\vec{x}, \;\; A = \begin{bmatrix} P_x(x_0, y_0) & P_y(x_0, y_0) \\ Q_x(x_0, y_0) & Q_y(x_0, y_0) \end{bmatrix}$$

at an equilibrium (x_0, y_0) predict its type (node, saddle, center, spiral point) and its stability?

Theorem 2.10. Let (x_0, y_0) be an isolated equilibrium for the nonlinear system, and let A be the Jacobian matrix for the linearization with $\det A \neq 0$. Then (x_0, y_0) is an equilibrium of the nonlinear system of the same type and stability as the equilibrium at the origin is of the linearization when

(i) the eigenvalues of A are real, either equal or distinct, and have the same sign (node),

(ii) the eigenvalues of A are real and have opposite sign (saddle), or

(iii) the eigenvalues are complex conjugate with nonzero real part (spiral).

Notice that the exception is when the linearization has a center: more information is needed to say that the linearization determines the type and stability of the nonlinear system near the equilibrium.

Example. A particle of mass $m = 1$ moving under the influence of a conservative force $f(x) = 1 - 3x^2$ is governed by

$$x' = y, \; y' = f(x) = 1 - 3x^2.$$

This mechanical system has a conversed total energy of

$$\frac{y^2}{2} + x^3 - x = E.$$

The critical points of the potential function $V(x) = x^3 - x$ (i.e., $-V'(x) = 1 - 3x^2$) are $x = \pm 1/\sqrt{3}$.

There are two equilibria of the nonlinear system; $(\pm 1/\sqrt{3}, 0)$.

With $P(x, y) = y$ and $Q(x, y) = 1 - 3x^2$, the Jacobian matrix is

$$A = \begin{bmatrix} P_x & P_y \\ Q_x & Q_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6x & 0 \end{bmatrix}.$$

The Jacobian of the linearization at $(1/\sqrt{3}, 0)$ is

$$A = \begin{bmatrix} 0 & 1 \\ -6/\sqrt{3} & 0 \end{bmatrix}$$

whose eigenvalues are pure imaginary,

This is the exception case and we can not use Theorem 2.10 to infer that the equilibrium $(1/\sqrt{3}, 0)$ is a center.

However, because of the conserved total energy, whose level curves give the orbits, we can conclude that $(1/\sqrt{3}, 0)$ is indeed a center.

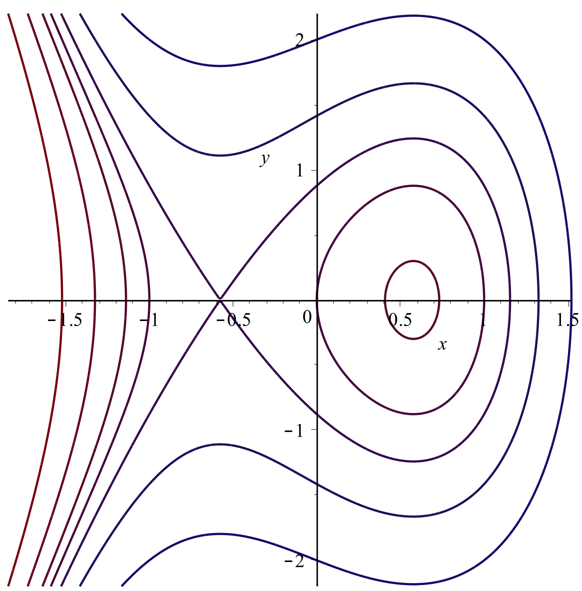
The Jacobian of the linearization at $(-1/\sqrt{3}, 0)$ is

$$A = \begin{bmatrix} 0 & 1 \\ 6/\sqrt{3} & 0 \end{bmatrix}$$

whose eigenvalues are real and of opposite sign.

By Theorem 2.10, we know that the equilibrium $(-1/\sqrt{3}, 0)$ is a saddle point.

We recall the level curves of the total energy from the last lecture.



When can we guarantee that an isolated equilibrium of a nonlinear system is locally asymptotically stable?

Theorem 2.11. If the equilibrium at the origin of the linearization is asymptotically stable, then the equilibrium (x_0, y_0) of the nonlinear system is locally asymptotically stable.

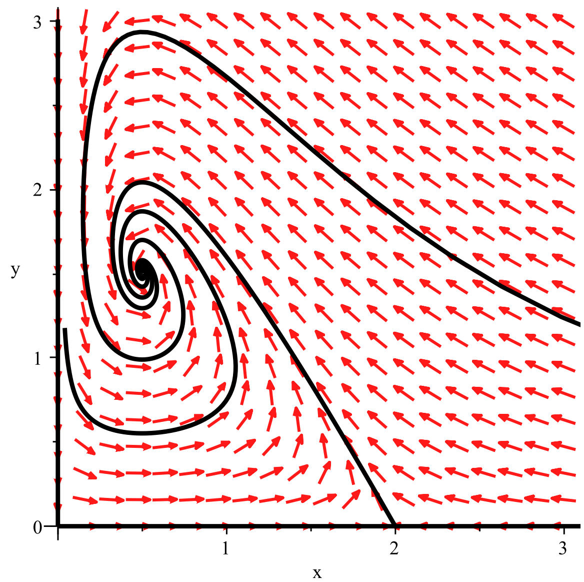
Example. The point $(1/2, 3/2)$ is an isolated equilibrium for the nonlinear system

$$x' = x(1 - 0.5x - 0.5y), \;\; y' = y(-0.25x + 0.5x).$$

The Jacobian of the linearization at this equilibrium is

$$A = \begin{bmatrix} -1/4 & -1/4 \\ 3/4 & 0 \end{bmatrix}.$$

The eigenvalues of A are $(-1 + i\sqrt{11})/8$; the equilibrium $(1/2, 3/2)$ is a locally asymptotically stable spiral point.



In sketching a phase portrait, once the equilibria have been found and analyzed, the next step is to determine if any periodic orbits exist.

There are some relatively easy conditions to verify by which a periodic orbit can not exist.

Theorem 2.17. For the system $x' = P(x, y)$, $y' = Q(x, y)$, if $P_x + Q_y$ is of one sign in a region of the phase plane, then the system cannot have a periodic orbit in that region.

Proof. Suppose that a region contains a periodic orbit $(x(t), y(t))$, $0 \le t \le T$ of the system.

Let Γ be the curve that the periodic orbit traces, and let R be the interior of Γ.

By Green's Theorem we have

$$\oint_\Gamma P dy - Q dx = \int\int_R (P_x + Q_y) dy dx \neq 0.$$

On the other hand,

$$\oint P dy - Q dx = \int_0^T (P y' - Q x') dt = \int_0^T (PQ - QP) dt = 0,$$

which is a contradiction. □

Theorem 2.18. A periodic orbit surrounds at least one equilibrium.

On the other hand, there are very few existence results for periodic orbits.

Theorem 2.19. Let R be a compact region of the phase plane that contains no critical points. If for a solution $(x(t), y(t))$, there exists t_0 such that $(x(t), y(t)) \in R$ for all $t \geq t_0$, then the solution is either a periodic orbit, or it spirals toward a periodic orbit.

Example. The only equilibrium of

$$x' = y,\ y' = -x + 0.2(1 - x^2)y$$

is at the origin.

The Jacobian of the linearization at this equilibrium is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0.2 \end{bmatrix}.$$

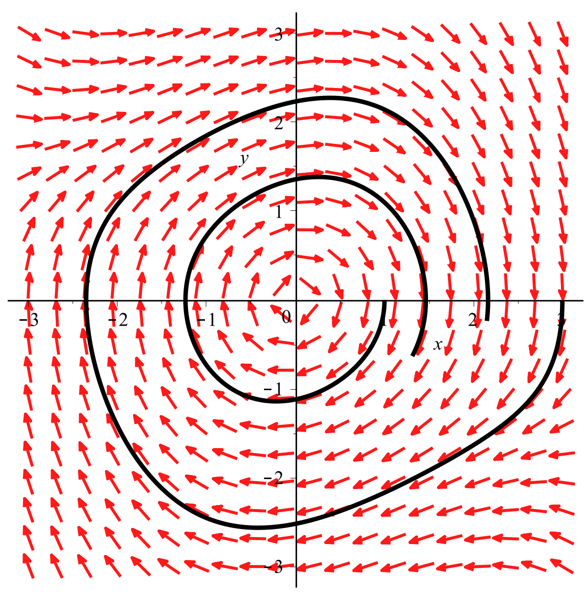
The eigenvalues of A are complex conjugates with positive real part, so the origin is unstable spiral point.

If there were to exist a periodic orbit, it would have to surround the equilibrium at the origin.

Since $P_x + Q_y = 1 - x^2$, no periodic orbit can exist in each of the strips $x < -1$, $|x| < 1$, and $x > 1$ in the phase plane.

So any periodic orbit, if it exists, must cross on of the two vertical lines $x = \pm 1$.

To form a compact region, follow the orbit that starts at $(3, 0)$ until it crosses the x-axis again, and follow the orbit that starts at $(1, 0)$ and follow it until it crosses the x axis.



A compact region R is form by these orbit segments and the pieces of the x-axis between each of their first and second intersections with the x-axis.

The orbit starting at $(3, 0)$ stays in R for all $t \geq 0$ and since this orbit is not periodic, it spirals toward a periodic orbit in R.

Thus there exists a periodic orbit for this nonlinear system and it crosses both $x = 1$ and $x = -1$.