# Math 521 Lecture #13
## §2.4: Bifurcations

There are many kinds of bifurcations that can occur in a parameter dependent system of nonlinear equations.

We will give examples of two of them.

### Example (Saddle-Node Bifurcation).

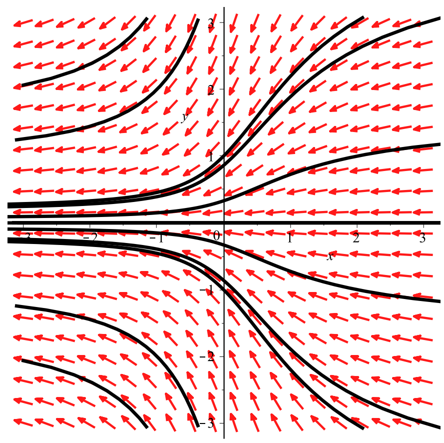
Consider the nonlinear system

$$x' = \mu - x^2,\ y' = -y.$$

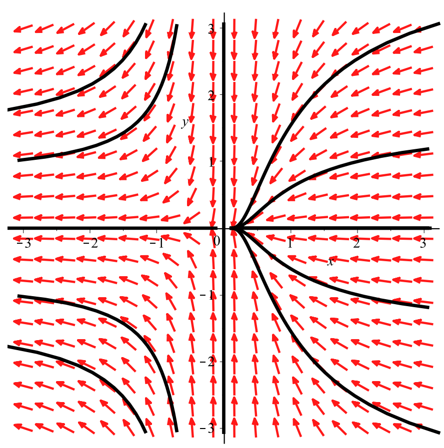
Each equation can be solved explicitly, but we will use the geometric analysis approach to see how the solutions change as $\mu$ is varied.

The phase line for $y' = -y$ shows that $y(t) \to 0$ for all choices of initial $y$ value.

For $\mu < 0$ we have that $x' < -\mu$, so there are no equilibria, and $x(t)$ decreases with bound.



For $\mu = 0$ we have one equilibrium at the origin with $x' > 0$ for $x < 0$ and $x > 0$ (so the equilibrium is unstable).

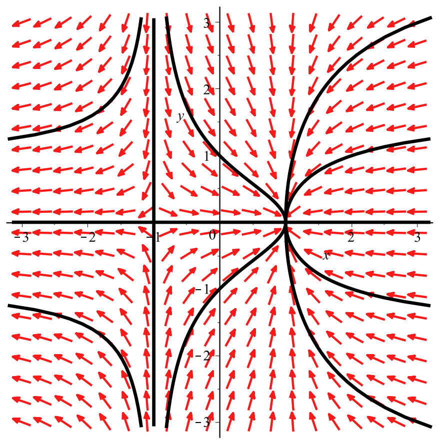


The Jacobian for the linearization at the origin (when $\mu = 0$) is diagonal with entries 0 and $-1$, which are the eigenvalues, and so the equilibrium is non-hyperbolic.

For $\mu > 0$ we have two equilibria $(x^*, 0)$ for $x^* = \pm\sqrt{\mu}$.

The Jacobian of the linearization at the equilibrium $(-\sqrt{\mu}, 0)$ is diagonal with entries $2\sqrt{\mu}$ and $-1$, and so it is saddle point.

The Jaocbian of the linearization at the equilibrium $(\sqrt{\mu}, 0)$ is diagonal with entries $-2\sqrt{\mu}$ and $-1$, and so it is a locally asymptotically stable node.



Hence at $\mu = 0$ we have a Saddle-Node Bifurcation.

## Example (Hopf Bifircation).
Consider the nonlinear system

$$x' = -y + x(\mu - x^2 - y^2), \ y' = x + y(\mu - x^2 - y^2).$$

The only equilibrium is at the origin.

The Jacobian matrix is

$$A = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}.$$

The characteristic equation of $A$ is

$$\lambda^2 - 2\mu + (\mu^2 + 1) = 0.$$

The eigenvalues are

$$\lambda = \frac{2\mu \pm \sqrt{4\mu^2 - 4(\mu^2 + 1)}}{2} = \mu \pm i.$$

For $\mu > 0$ the equilibrium is an unstable spiral point,

For $\mu = 0$ the equilibrium is a linear center.

For $\mu < 0$ the equilibrium is a locally asymptotically stable spiral point.

What does the phase portrait look like when $\mu = 0$? And, what happens to the phase portrait as $\mu$ passes through $\mu = 0$ from negative to positive?

These can be explicitly answered by transforming the equations into polar coordinates.

With $x = r\cos\theta$, $y = r\sin\theta$ we have

$$\begin{aligned} xx' + yy' &= r\cos\theta(r'\cos\theta - r\theta'\sin\theta) + r\sin\theta(r'\sin\theta + r\theta'\cos\theta) \\ &= rr'\cos^2\theta - r^2\theta^2\cos\theta\sin\theta + rr'\sin^2\theta + r^2\sin\theta\cos\theta \\ &= rr' \end{aligned}$$

and

$$\begin{aligned} xy' - yx' &= r\cos\theta(r'\sin\theta + r\theta'\cos\theta) - r\sin\theta(r'\cos\theta - r\theta'\sin\theta) \\ &= rr'\cos\theta\sin\theta + r^2\theta'\cos^2\theta - rr'\sin\theta\cos\theta + r^2\theta'\sin^2\theta \\ &= r^2\theta'. \end{aligned}$$

Using the nonlinear system we have

$$rr' = xx' + yy' = -xy + x^2(\mu - x^2 - y^2) + xy + y^2(\mu - x^2 - y^2) = r^2(\mu - r^2)$$

and

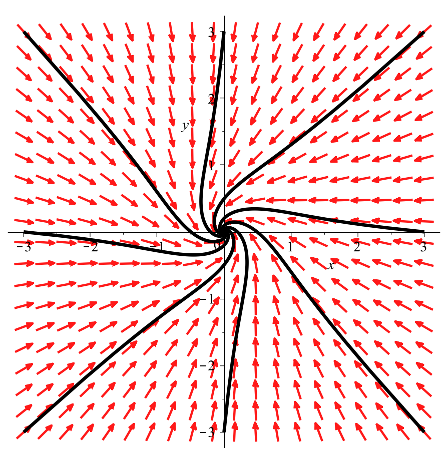
$$r^2\theta' = xy' - yx' = x^2 + xy(\mu - x^2 - y^2) + y^2 - xy(\mu - x^2 - y^2).$$

The nonlinear system in polar coordinates is

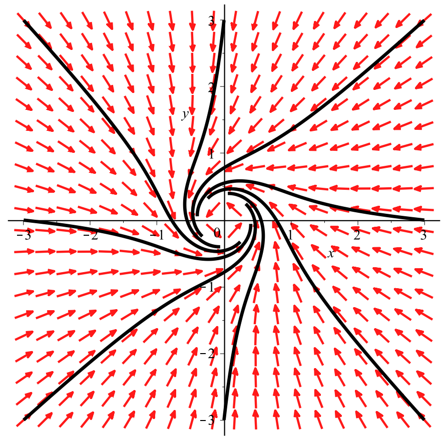
$$r' = r(\mu - r^2),\ \ \theta' = 1.$$

To understand the nonlinear system, we apply the geometric analysis approach to $r' = r(\mu - r^2)$ (although we could solve this separable equation as the book does).

For $\mu < 0$ we have that $r' < 0$ for all $r > 0$, so that all non-equilibrium solutions in the $xy$-plane tend to the origin (which in the linearization is locally asymptotically stable equilibrium).



For $\mu = 0$, we have that $r' < 0$ for all $r > 0$, so that all non-equilibrium solutions in the $xy$-plane tend to the origin, and so the origin is a not a center (although the linearization is a center).



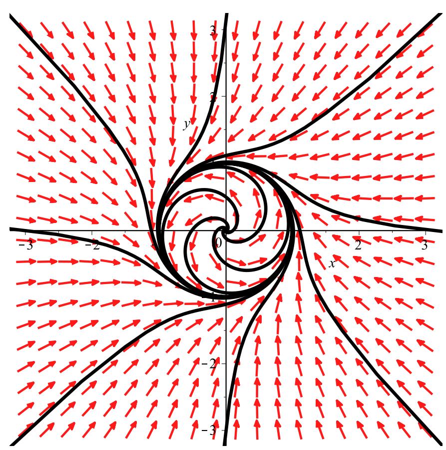
For each fixed $\mu > 0$, the constant function $r(t) = \sqrt{\mu} > 0$ is a solution of $r' = r(\mu - r^2)$.

Coupled with $\theta' = 1$, we get a periodic solution $r(t) = \sqrt{\mu}$, $\theta(t) = t + \theta_0$ in the $xy$-plane.

The sign of $r' = r(\mu - r^2)$ when $0 < r < \sqrt{\mu}$ is positive, so solutions starting near the origin in the $xy$-plane spiral away from it (the origin is an unstable spiral point), and move towards the periodic orbit $r = \sqrt{\mu}$.

The sign of $r' = r(\mu - r^2)$ when $r > \sqrt{\mu}$ is negative, so solutions starting with large $r$ value in the $xy$-plane spiral towards the period orbit $r = \sqrt{\mu}$.

The periodic orbit $r = \sqrt{\mu}$ is called a **stable limit cycle**.



The appearance of a stable (unstable) limit cycle from a stable (unstable) equilibrium as the parameter varies is known as a Hopf bifurcation.