Math 521 Lecture #15
§3.1,3.1.1: Regular Perturbation Theory

The basic idea of perturbation theory is to find analytic approximations to solutions of equations.

Consider the equation $F(t, y, y', y'', \ldots, \epsilon) = 0$, $t \in I$, where $\epsilon \ll 1$.

A **perturbation series** is an analytic guess for a solution of the form

$$y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots.$$

The basic idea of the **regular perturbation** method is to substitute this guess into the equation and solve for $y_0(t)$, $y_1(t)$, $y_2(t)$, etc.

The first few terms of a perturbation series are called a **perturbation solution** or approximation.

We call $y_0(t)$ the **leading order** term of the perturbation series.

If this method is successful, then $y_0(t)$ should be a solution of the unperturbed equation $F(t, y, y', y'', \ldots, 0) = 0$.

Example 3.1. We apply the perturbation method to approximate the roots of

$$x^2 + 2\epsilon x - 3 = 0.$$

We can check the approximate solutions against the exact solutions, since we know how to the exact solutions in this case.

We substitute the perturbation series

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

into the equation to get

$$(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^2 + 2\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) - 3 = 0.$$

Expanding and collecting the coefficients of like powers of ϵ gives us

$$x_0^2 - 3 + 2x_0(x_1 + 1)\epsilon + (x_1^2 + 2x_0 x_2 + 2x_1)\epsilon^2 + \cdots = 0.$$

Since this power series in ϵ equals zero, the coefficients of each power of ϵ must be zero, which gives us

$$x_0^2 = 3,\ x_1 = -1,\ x_1^2 + 2x_0 x_2 + 2x_1 = 0, \ldots.$$

These imply that

$$x_0 = \pm\sqrt{3},\ x_1 = -1,\ x_2 = \pm\frac{1}{2\sqrt{3}}, \cdot.$$

From these we get two approximate solutions

$$x = -\sqrt{3} - \epsilon + \frac{1}{2\sqrt{3}}\epsilon^2 + \cdots,\ x = -\sqrt{3} - \epsilon - \frac{1}{2\sqrt{3}}\epsilon^2 + \cdots.$$

We have three-term perturbation approximations of the two roots of the quadratic polynomial.

The exact roots are

$$x = \frac{-2\epsilon \pm \sqrt{4\epsilon^2 + 12}}{2} = -\epsilon \pm \sqrt{3 + \epsilon^2}.$$

Using the binomial formula

$$(1 + x)^p = 1 + px + \frac{p(p-1}{2}x^2 + \cdots,$$

we obtain

$$\sqrt{3 + \epsilon^2} = \sqrt{3}\left(1 + \frac{\epsilon^2}{\sqrt{3}}\right)^{1/2} = \sqrt{3}\left(1 + \frac{\epsilon^2}{6} + \cdots\right).$$

Thus the exact solutions are

$$x = \pm\sqrt{3} - \epsilon \pm \frac{1}{2\sqrt{3}}\epsilon^2 + \cdots$$

which is what the perturbation method obtained.

Now we apply the perturbation method to a differential equation.

Example (§1.3.1). Suppose a body of mass m, initially with a velocity of V_0, moves in a resistive medium along a straight line.

When the velocity of the body is $v = v(t)$, the force of the resistance is $-av + bv^2$ for positive constants a and b.

We assume that $b \ll a$, meaning that b is much smaller than b.

By Newton's second law, the velocity of the body is governed by the IVP

$$m\frac{dv}{dt} = -av + bv^2,\ v(0) = V_0.$$

We have four parameters $[m] = M$, $[a] = MT^{-1}$, $[b] = ML^{-1}$, and $[V_0] = LT^{-1}$, and so we scale time and velocity to non-dimensionalize the IVP.

The scale for the velocity it is maximum value V_0 which occurs at the start $t = 0$.

If b were 0, then the velocity would act like the solution $V_0e^{-at/m}$ of $v' = -v$, $v(0) = V_0$.

So a scale for the time is m/a.

With

$$y = \frac{v}{V_0},\ \tau = \frac{t}{m/a} = \frac{at}{m},$$

the IVP becomes

$$\frac{maV_0}{m}\frac{dy}{d\tau} = \frac{dv}{dt} = -aV_0y + bV_0^2y^2,\ y(0) = \frac{V_0}{V_0} = 1.$$

The ODE becomes

$$\frac{dy}{dt} = -y + \epsilon y^2$$

where

$$\epsilon = \frac{bV_0}{a} \ll 1.$$

The ODE is a Bernoulli equation that can be solved by the change of variable $w = y^{-1}$ to give the solution of the IVP as

$$y(t) = \frac{e^{-t}}{1 + \epsilon(e^{-t} - 1)}.$$

Expanding this solution as a Taylor series in ϵ we get

$$y(t) = e^{-t} + \epsilon(e^{-t} - e^{-2t}) + \epsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \cdots.$$

Notice that the leading term e^{-t} is the solution of the unperturbed $y' = -y$, $y(0) = 1$, and provided an analytic approximation to the solution of the perturbed $y' = -y + \epsilon y^2$, $y(0) = 0$ for small ϵ.

To see that this is so, we form the perturbation series

$$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y - 2(t) + \cdots$$

and substitute this guess into the ODE to get

$$y_0' + \epsilon y_1' + \epsilon^2 y_2'' + \cdots = -(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots) + \epsilon(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots)^2.$$

Expanding and collecting like terms gives the sequence of ODEs

$$y_0' = -y_0, \quad y_1' = -y_1 + y_0^2, \quad y_2' = -y_2 + 2y_0 y_1, \text{ etc.}$$

Applying the initial condition $y(0) = 1$ to the perturbation series gives

$$y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + \cdots = 1$$

which implies that

$$y(0) = 1, \quad y_1(0) = 0, \quad y_2(0) = 0, \text{ etc.}$$

We obtain a recursive set of linear IVPs for the coefficient functions in the perturbation series which we solve to get

$$y_0 = e^{-t}, \quad y_1 = e^{-t} - e^{-2t}, \quad y_2 = e^{-t} - 2e^{-2t} + e^{-3t}, \text{ etc.}$$

A three-term perturbation solution is

$$y = e^{-t} + \epsilon(e^{-t} - e^{-2t}) + \epsilon^2(e^{-t} - 2e^{-2t} + e^{-3t})$$

which is the degree two Taylor polynomial coming from the Taylor series expansion of the exact solution.

We think of the ϵ and ϵ^2 terms as first- and second-order corrections to the leading order term $y_0 = e^{-t}$.