Math 521 Lecture #16
§3.1.2: Nonlinear Oscillations

We saw last time that the perturbation method gave approximations of solutions that compared favorably with the exact solutions.

We apply the perturbation method to another nonlinear problem.

Example. Consider an undamped mass-spring system for an object of mass m with displacement y from equilibrium,.

We assume that the restoring force of the spring is the nonlinear $ky + ay^3$ for positive constants k and a (characterizing the stiffness of the spring).

We further assume that $a \ll k$, so that the nonlinear part of the restoring force is small when compared with the linear part.

The object is released from a positive displacement A from equilibrium.

The IVP that models the displacement $y = y(\tau)$ of the object in this nonlinear mass-spring system is

$$m\frac{d^2y}{dt^2} = -ky - ay^3, \ \tau > 0,$$
$$y(0) = A, \ y'(0) = 0.$$

Without damping we reasonably expect that non-equilibrium solutions should be periodic, that the equilibrium at the origin is a nonlinear center.

However, the presence of the nonlinearity y^3 means that the problem cannot be solved exactly to confirm this.

But, because $a \ll k$, a perturbation method is appropriate to find an an approximation to a periodic solution.

We first non-dimensionalize the problem.

We dimensions of the four parameters in the problem are $[k] = MT^{-2}$, $[a] = ML^{-2}T^{-2}$, $[m] = M$, and $[A] = L$.

We use the initial displacement A to scale y:

$$u = \frac{y}{A}.$$

For a time scale we look for one that permits us to neglect the "small" term ay^3.

By so doing we get $my'' = -ky$ which has periodic solutions of the form $\cos\sqrt{kt/m}$ with a period proportional to $\sqrt{m/k}$.

We use this periodic to scale time:

$$t = \frac{\tau}{\sqrt{m/k}}.$$

With this scaling of t and y, the ODE becomes

$$m\left(\frac{kA}{m}\right)\frac{d^2u}{dt^2} = m\frac{dy}{dt} = -kAu - aA^3u^3.$$

For a dimensionless parameter $\epsilon = aA^2/k$ we have the *Duffing equation*

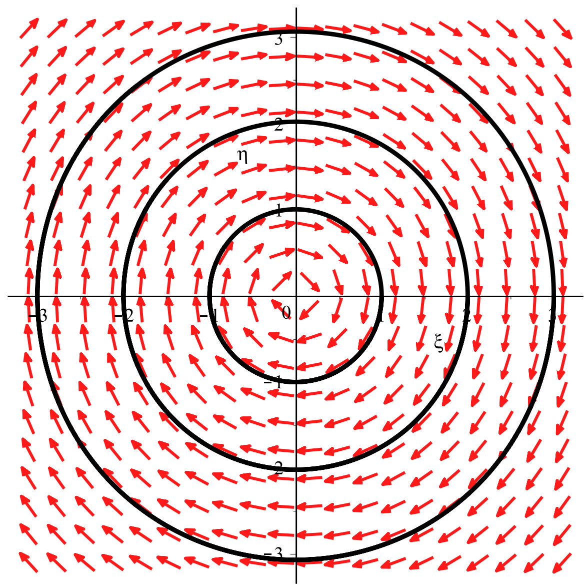
$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = 0.$$

The initial conditions become

$$u(0) = \frac{A}{A} = 1,\ \frac{du}{dt}(0) = 0.$$

Assuming that ϵ is small means that $aA^2 \ll k$ and not just $a \ll k$.

Here is the phase portrait of the Duffing equation when $\epsilon = 0.01$ in the variables $\xi = u$ and $\eta = u'$, i.e., $\xi' = \eta$, and $\eta' = -\xi - \epsilon\xi^3$.



It appears numerically that the equilibrium at the origin is a nonlinear center: every non-equilibrium solution is periodic.

The level sets (circles) of $z = \xi^2 + \eta^2$ are "almost" solutions of the ODE because

$$\frac{d}{dt}\big(\xi^2 + \eta^2\big) = 2\xi\xi' + 2\eta\eta' = 2\big(\xi\eta - \eta(\xi + \epsilon\xi^3)\big) = -2\epsilon\xi^3\eta \approx 0.$$

The perturbation guess for a periodic solution has the form

$$u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \cdots$$

for functions u_0, u_1, u_2, etc., to be determined.

Substitution of the perturbation series into the IVP gives

$$(u_0'' + \epsilon u_1'' + \epsilon^2 u_2'' + \cdots) + (u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots) + \epsilon(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots)^3 = 0,$$
$$1 = u_0(0) + \epsilon u_1(0) + \epsilon^2 u_2(0) + \cdots, \ 0 = u'(0) = u_0'(0) + \epsilon u_1'(0) + \epsilon^2 u_2'(0) + \cdots.$$

The first three linear second-order IVPS in the sequence of linear second-order IVPS from this are

$$u_0'' + u_0 = 0, \ u_0(0) = 1, \ u_0'(0) = 0,$$
$$u_1'' + u_1 = -u_0^3, \ u_1(0) = 0, \ u_1'(0) = 0,$$
$$u_2'' + u_2 = -3u_0^2 u_1, \ u_2(0) = 0. \ u_2'(0) = 0.$$

The first one gives

$$u_0(t) = \cos t,$$

which matches the solution of the unperturbed problem.

Using the solution of the first IVP, the second IVP is

$$u_1'' + u_1 = -\cos^3 t, \ u_1(0) = 0, \ u_1'(0) = 0.$$

To solve this one we need the trigonometry identity

$$\cos 3t = 4\cos^3 t - 3\cos t$$

so that the linear second-order ODE becomes

$$u_1'' + u_1 = -\frac{3\cos t + \cos 3t}{4}.$$

The general solution of the homogeneous part is $c_1 \cos t + c_2 \sin t$.

For a particular solution we use the Method of Undetermined Coefficients to guess the particular solution as

$$u_p = C\cos 3t + Dt\cos t + Et\sin t.$$

With

$$u_p' = -3C\sin 3t + D(\cos t - t\sin t) + E(\sin t + t\cos t)$$

and

$$u_p'' = -9C\cos 3t + D(-2\sin t - t\cos t) + E(2\cos t - t\sin t)$$

we have that the undetermined coefficients in u_p satisfy

$$-8C = -\frac{1}{4}, \ -2D = 0, \ 2E = -\frac{3}{4}.$$

Thus

$$C = \frac{1}{32}, \ D = 0, \ E = -\frac{3}{8}.$$

With the general solution being

$$u_2 = c_1 \cos t + c_2 \sin t + \frac{1}{32}\cos 3t - \frac{3}{8} t\sin t,$$

the initial conditions $u_2(0) = 0$ and $u_2'(0) = 0$ imply that c_1 and c_2 satisfy

$$0 = c_1 + \frac{1}{32}, \;\; 0 = c_2.$$

The solution of the IVP for u_2 is

$$u_2 = \frac{\cos 3t - \cos t}{32} - \frac{3}{8} t \sin t.$$

The two-term approximation for u then takes the form

$$u_a = \cos t + \epsilon \left[\frac{\cos 3t - \cos t}{32} - \frac{3}{8} t \sin t \right].$$

The leading-order term is periodic, but the correction term is not.

Even for very small ϵ, the second term will eventually get large because the **secular term** $-(3/8)t \sin t$ is unbounded as $t \to \infty$.

This disagrees with the numerically generated phase portrait of the system.

This means that on the interval $[0, \infty)$ the two-term approximation is not a uniform approximation.

Adding in higher order terms to the approximation doesn't negate the secular effect of the second term.

The best we can conclude is that the two-term (or higher order term) approximation is uniform on a given finite length interval $[0, T]$ for sufficiently small ϵ.