Math 521 Lecture #17
§3.1.3: The Poincaré-Lindstedt Method

Recall that the application of the perturbation method to finding an approximation of the periodic solution of the Duffing equation

$$\frac{d^2u}{dt^2} + u + \epsilon u^3 = 0,\ u(0) = 1,\ u'(0) = 0$$

gave the two-term approximation

$$u_a = \cos t + \epsilon\left[\frac{1}{32}\big(\cos 3t - \cos t\big) - \frac{3}{8}t\sin t\right]$$

that contains a secular term that made the approximation invalid unless t was restricted to a finite interval.

The correction term also does not account for the expected small change in the period of the oscillation as the parameter ϵ is perturbed from 0 to something small and positive.

Even if the correction term did not have the secular term, the approximation would fail to be uniform on $(0,\infty)$ because the error in the period would be magnified over time putting the exact solution and the approximate solution "out of phase."

The idea of Poincaré and Lindstedt is to distort the time scale in the perturbation series

$$u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \cdots$$

by the non-dimensional

$$\tau = \omega t$$

for

$$\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots.$$

The idea is to choose ω_1, ω_2, etc., to avoid secular terms in u_1, u_2, etc.

With the scaling of the time variable we have

$$\frac{du}{dt} = \frac{du}{d\tau}\frac{d\tau}{dt} = \omega\frac{du}{d\tau}.$$

So in the new time variable, the IVP becomes

$$\omega^2 u'' + u + \epsilon u^3 = 0,\ u(0) = 1,\ u'(0) = 0,$$

where the prime denote differentiation with respect to the time variable τ.

Substitution of the perturbation series with the distorted time scale into the IVP gives

$$\begin{aligned}
&(1 + \epsilon w_1 + \epsilon^2\omega_2 + \cdots)^2(u_0'' + \epsilon u_1'' + \epsilon^2 u_2'' + \cdots)\\
&\qquad + (u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots) + \epsilon(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots)^3 = 0,\\
&1 = u_0(0) + \epsilon u_1(0) + \epsilon^2 u_2(0) + \cdots,\\
&0 = u_0'(0) + \epsilon u_1'(0) + \epsilon^2 u_2'(0) + \cdots.
\end{aligned}$$

Collecting the coefficients of like powers of ϵ gives a sequence of linear second-order IVPS, the first three of which are

$$\begin{aligned}
&u_0'' + u_0 = 0,\ u_0(0) = 1,\ u_0'(0) = 0,\\
&u_1'' + u_1 = -2\omega_1 u_0'' - u_0^3,\ u_1(0) = 0,\ u_1'(0),\\
&u_2'' + u_2 = -(\omega_1^2 + 2\omega_2)u_0'' - 2\omega_1 u_1'' - 3u_0^2 u_1,\ u_2(0) = 0,\ u_2'(0) = 0.
\end{aligned}$$

The solution of the first linear second-order IVP is

$$u_0(\tau) = \cos\tau.$$

The second linear second-order IVP becomes

$$u_1'' + u_1 = 2\omega_1\cos\tau - \cos^3\tau,\ u_1(0) = 0,\ u_1'(0) = 0.$$

Using the trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

the ODE becomes

$$u_1'' + u_1 = 2\omega_1\cos\tau - \frac{\cos 3\tau + 3\cos\tau}{4} = \left(2\omega_1 - \frac{3}{4}\right)\cos\tau - \frac{\cos 3\tau}{4}.$$

The presence of $\cos\tau$ in the nonhomogeneous term leads to the secular term in the solution.

To remove $\cos\tau$ we choose

$$\omega_1 = \frac{3}{8}.$$

With this choice, a particular solution of

$$u_1'' + u_1 = -\frac{\cos 3\tau}{4}$$

has the form

$$u_p = A\cos 3\tau.$$

The undetermined coefficient A satisfies

$$-9A + A = -\frac{1}{4},$$

so that $A = 1/32$.

The general solution of

$$u_1'' + u_1 = -\frac{\cos 3\tau}{4}$$

is

$$u_1 = c_1\cos\tau + c_2\sin\tau + \frac{1}{32}\cos 3\tau.$$

The solution of IVP

$$u_1'' + u_1 = -\frac{\cos 3\tau}{4},\ \ u_1(0) = 0,\ \ u_1'(0) = 0$$

is then

$$u_1(t) = \frac{\cos 3\tau - \cos\tau}{32}.$$

We obtain a two-term approximation of the periodic solution,

$$u_a = \cos\tau + \epsilon\left[\frac{\cos 3\tau - \cos\tau}{32}\right]$$

for the distorted time

$$\tau = \omega t = t + \epsilon\omega_1 t + \cdots = t + \epsilon\frac{3t}{8} + \cdots.$$

This approximation is uniformly valid on $(0, \infty)$ for sufficiently small ϵ.