Math 521 Lecture #18
§3.1.4: Asymptotic Analysis

We have found previously what we called good (or uniform) approximations to exact solutions of IVPs.

To what extend do these good approximations really approximate exact solutions?

We shall quantity this notion of good approximation in the context of two order relations.

Definitions. Let $f(\epsilon)$ and $g(\epsilon)$ be defined in some neighbourhood (or punctured neighbourhood) of $\epsilon = 0$.

(We write

$$f(\epsilon) = o(g(\epsilon)) \text{ as } \epsilon \to 0,$$

and say f is **little oh** of g if

$$\lim_{\epsilon\to 0}\left|\frac{f(\epsilon)}{g(\epsilon)}\right| = 0.$$

We write

$$f(\epsilon) = O(g(\epsilon))$$

and say f is **big oh** of g if there exists $M > 0$ such that

$$|f(\epsilon)| \le M|g(\epsilon)|$$

for all ϵ in some neighbourhood (or punctured neighbourhood) of 0.

In both of these, we can replace $\epsilon \to 0$ with $\epsilon \to \epsilon_0$ for ϵ_0 finite or infinite, and/or one-sided limits.

The comparison function g is called a **gauge function**.

Common gauge functions are $g(\epsilon) = \epsilon^n$ and $g(\epsilon) = \epsilon^n(\ln\epsilon)^m$ for real numbers m and n.

To say $f(\epsilon) = O(1)$ means that f is bounded in a neighbourhood of $\epsilon = 0$.

To say $f(\epsilon) = o(1)$ means that $f(\epsilon) \to 0$ as $\epsilon \to 0$.

When $f = o(g)$, then f goes to 0 faster than g does as $\epsilon \to 0$, and we write $f(\epsilon) \ll g(\epsilon)$.

Example 3.2. We can verify that $\epsilon^2\ln\epsilon = o(\epsilon)$ as $\epsilon \to 0^+$ by L'Hôpital's rule:

$$\lim_{\epsilon\to 0^+}\frac{\epsilon^2\ln\epsilon}{\epsilon} = \lim_{\epsilon\to 0^+}\frac{\ln\epsilon}{1/\epsilon} = \lim_{\epsilon\to 0^+}\frac{1/\epsilon}{-1/\epsilon^2} = 0.$$

Example 3.3. Does $\sin\epsilon = O(\epsilon)$ as $\epsilon \to 0$?

By the Mean Value Theorem applied to $\sin\theta$ on the interval $[0,\epsilon]$, there exists $c \in (0,\epsilon)$ such that

$$\frac{\sin\epsilon}{\epsilon} = \frac{\sin\epsilon - \sin 0}{\epsilon - 0} = \cos c.$$

Since $|\cos\theta|$ is bounded above by $M = 1$, we have that

$$\left|\frac{\sin\epsilon}{\epsilon}\right| \le 1,$$

and so $\sin\epsilon = O(\epsilon)$ as $\epsilon \to 0$.

Another argument for $\sin\epsilon = O(\epsilon)$ comes from the well-known limit

$$\lim_{\epsilon\to 0} \frac{\sin\epsilon}{\epsilon} = 1.$$

Because the limit exists, the function $\epsilon^{-1}\sin\epsilon$ is bounded in a neighbourhood of $\epsilon = 0$.

So there exists $M > 0$ such that

$$\left|\frac{\sin\epsilon}{\epsilon}\right| \le M$$

for ϵ in some punctured neighbourhood of 0, and this gives the big oh property.

We can extend the order relations to functions of the form $f(t,\epsilon)$.

Definitions. Let $f(t,\epsilon)$ and $g(t,\epsilon)$ be defined for t in some interval I and for ϵ is a (punctured) neighbourhood of 0.

We say $f(t,\epsilon) = o(g(t,\epsilon)$ pointwise on I as $\epsilon \to 0$ if

$$\lim_{\epsilon\to 0}\left|\frac{f(t,\epsilon)}{g(t,\epsilon)}\right| = 0$$

We say $f(t,\epsilon) = O(g(t,\epsilon))$ pointwise on I as $\epsilon \to 0$ if for each $t \in I$ there exists a positive $M(t)$ such that

$$|f(t,\epsilon)| \le M(t)|g(t,\epsilon)|$$

for all ϵ in some (punctured) neighbourhood of 0.

We recall the notion of uniform convergence in the context of functions of the form $h(t,\epsilon)$.

Let $h(t,\epsilon)$ be defined for t in some interval I, and ϵ in a (punctured) neighbourhood of 0.

We write

$$\lim_{\epsilon\to 0} h(t,\epsilon) = 0 \text{ uniformly on } I$$

and say h converges uniformly to 0 on I as $\epsilon \to 0$, if for every $\eta > 0$ there exist $\epsilon_0 > 0$ such that

$$|h(t,\epsilon)| < \eta$$

for all $t \in I$ whenever $|\epsilon| < \epsilon_0$.

Typically to prove that $h(t,\epsilon) \to 0$ uniformly on I, we find a function $H(\epsilon)$ such that $|h(t,\epsilon)| \le H(\epsilon)$ for all $t \in I$ and $H(\epsilon) \to 0$ as $\epsilon \to 0$.

We say that $f(t,\epsilon) = o(g(t,\epsilon))$ uniformly on I as $\epsilon \to 0$ if the limit is uniform on I.

We say that $f(t,\epsilon) = O(g(t,\epsilon))$ uniformly on I is there exists a bounded positive function $M : I \to \mathbb{R}$ such that

$$|f(t,\epsilon)| \le M(t)|g(t,\epsilon)|$$

for all ϵ in a (punctured) neighbourhood of 0.

A function $y_a(t,\epsilon)$ is a **uniformly valid asymptotic approximation** to a function $y(t,\epsilon)$ on an interval I as $\epsilon \to 0$ if the error $E(t,\epsilon) = y(t,\epsilon) - y_a(t,\epsilon)$ converges to 0 uniformly on I as $\epsilon \to 0$.

We use the little oh and big oh notions to quantify the rate at which the error goes to zero of a uniformly valid asymptotic approximation as in $E(t,\epsilon) = o(\epsilon^n)$ uniformly on I as $\epsilon \to 0$, or as in $E(t,\epsilon) = O(\epsilon^n)$ uniformly on I as $\epsilon \to 0$, for some n.

Example 3.4. The first three terms of the Taylor series expansion of

$$y(t,\epsilon) = e^{-t\epsilon},\ t > 0,\ \epsilon \ll 1$$

give an approximation

$$y_a(t,\epsilon) = 1 - t\epsilon + \frac{t^2\epsilon^2}{2}.$$

The error is

$$E(t,\epsilon) = e^{-t\epsilon} - 1 + t\epsilon - \frac{t^2\epsilon^2}{2} = -\frac{t^3\epsilon^3}{3!} + \frac{t^4\epsilon^4}{4!} - \cdots.$$

We have that

$$|E(t,\epsilon)| \le \frac{t^3\epsilon^3}{3!} + \frac{t^4\epsilon^4}{4!} + \cdots.$$

For a fixed $T > 0$, if we let $M = \exp(T)$, then for all $t \in [0,T]$ and all $\epsilon < 1$ we have

$$\left|\frac{E(t,\epsilon)}{\epsilon^3}\right| = \frac{t^3}{3!} + \frac{t^4\epsilon}{4!} + \cdots \le \frac{T^3}{3!} + \frac{T^4}{4!} + \cdots \le M.$$

Thus $E(t,\epsilon) = O(\epsilon^3)$ uniformly on $I = [0,T]$ as $\epsilon \to 0$.

However we do not have $E(t,\epsilon) = O(\epsilon^3)$ uniformly on $[0,\infty)$ as $\epsilon \to 0$ because for any $\epsilon > 0$ we can choose $t = \epsilon^{-1}$ for which

$$E(t,\epsilon) = e^{-1} - 1 + 1 - \frac{1}{2} = e^{-1} - \frac{1}{2}$$

which is not small.
The difficulty in directly applying the order relations to approximate solutions of a differential equation is when the exact solutions of the differential equation are not known.

We can, however, indirectly make a comparison.

For an approximate solution $y_a(t,\epsilon)$ of a differential equation $F(t,y,y',y'',\ldots,\epsilon) = 0$ we form the residual error

$$r(t,\epsilon) = F(t, y_a'(t,\epsilon), y_a''(t,\epsilon), \ldots, \epsilon).$$

We say that the approximate solution $y_a(t,\epsilon)$ satisfies the differential equation uniformly for $t \in I$ as $\epsilon \to 0$ if

$$r(t,\epsilon) \to 0$$

uniformly on I as $\epsilon \to 0$.

Example 3.6. Consider the IVP

$$y'' + (y')^2 + \epsilon y = 0, \ y(0) = 0, \ y'(0) = 1$$

for $t > 0$ and $\epsilon \ll 1$.

Substitution of the perturbation series $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$ into the ODE gives the IVP

$$y_0'' + (y_0')^2 = 0, \ y_0(0) = 0, \ y_0'(0) = 1.$$

Solving this gives

$$y_0(t) = \ln(t+1),$$

so that the residual error of y_0 is

$$r(t,\epsilon) = y_0'' + (y_0')^2 - \epsilon y_0 = \epsilon \ln(t+1).$$

Thus $r(t,\epsilon) = O(\epsilon)$ as $\epsilon \to 0$ uniformly on $I = [0,T]$ for finite T because

$$|\epsilon \ln(t+1)| \leq \epsilon \ln(T+1),$$

but not uniformly on $[0,\infty)$ because $\ln(t+1) \to \infty$ as $t \to \infty$.

Asymptotical Expansions. Application of the regular perturbation method produces an *expansion*

$$y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots$$

of a solution, the first few terms of which give an approximate solution.

A general expansion has the form

$$\phi_0(\epsilon) y_0(t) + \phi_1(\epsilon) y_1 + \phi_2(\epsilon) y_2 + \cdots$$

where $\{\phi_n\}_{n=0}^{\infty}$ is a sequence of functions, such as $\phi_n(\epsilon) = (\epsilon^r)^n$ for some $r > 0$.

To formalize what is meant by *asymptotic* in such an expansion we consider how the functions $g_n(t,\epsilon) = \phi_n(t) y_n(t)$ are related to each other.

A sequence of gauge functions $\{g_n(t,\epsilon)\}$ is called an **asymptotic sequence** as $\epsilon \to 0$ for $t \in I$ if for each $n = 0, 1, 2, \ldots$ we have

$$g_{n+1}(t,\epsilon) = o(g_n(t,\epsilon)) \text{ as } \epsilon \to 0.$$

For a function $y(t,\epsilon)$ and an asymptotic sequence $\{g_n(t,\epsilon)\}$, the *formal* series

$$\sum_{n=0}^{\infty} a_n g_n(t,\epsilon)$$

for constants a_n, is said to be an **asymptotic expansion** of $y(t,\epsilon)$ as $\epsilon \to 0$ if for every $N \geq 0$ we have

$$y(t,\epsilon) - \sum_{n=0}^{N} a_n g_n(t,\epsilon) = o(g_N(t,\epsilon)) \text{ as } \epsilon \to 0.$$

If the limits cited here are uniform on I, then we say uniform asymptotic sequence and uniform asymptotic expansion.