Math 521 Lecture #19
§3.2.1,3.2.2: Singular Perturbation

We explore through two examples how we modify the regular perturbation method when it fails.

Such modifications are part of the world of singular perturbation methods.

Example §3.2.1.
Consider solving

$$\epsilon x^2 + 2x + 1 = 0,\ 0 < \epsilon \ll 1$$

through a perturbation series

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots.$$

The only solution of the unperturbed equation is $2x + 1 = 0$ is $x = -1/2$.

Substituting the perturbation series into the perturbed equation gives

$$\epsilon(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)^2 + 2(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) + 1 = 0.$$

Equating coefficients of like powers of ϵ gives

$$\begin{aligned} &2x_0 + 1 = 0\ \ [\text{the unperturbed equation}] \\ &x_0^2 + 2x_1 = 0, \\ &2x_1 x_0 + 2x_2 = 0,\ \text{etc.} \end{aligned}$$

Solving for x_0, x_1, and x_2 etc., gives a single perturbation solution

$$x = -\frac{1}{2} - \frac{\epsilon}{8} - \frac{\epsilon^2}{16} + \cdots.$$

Where is the other solution of the perturbed quadratic equation?

The regular perturbation series assumed a leading-order term of order unity (the one solution $-1/2$ of the unperturbed linear equation).

The second solution of the perturbed quadratic equation could have a larger or smaller order than that of $-1/2$.

When we compare the orders of the terms ϵx^2, $2x$, and 1 in the perturbed quadratic equation, we see that for $x \approx -1/2$, the terms $2x$ and 1 have the same order, but the term ϵx^2 is very small and can safely be ignored.

The missing root of the perturbed equation could be small, so that ϵx^2 and $2x$ are of small order.

But this is impossible because $\epsilon x^2 + 2x$ is then not of the same order as that of 1.

So the second root of perturbed quadratic equation is large (of order bigger than 1).

In this case ϵx^2 and $2x$ are both large compared with 1.

Ignoring the "small" term of 1 in the perturbed equation gives

$$\epsilon x^2 + 2x = 0$$

which for a large x solves to give $x = 2/\epsilon$.

The order of this x is $O(1/\epsilon)$ as $\epsilon \to 0$ because

$$\left|\frac{x}{1/\epsilon}\right| = \left|\frac{2/\epsilon}{1/\epsilon}\right| = 2.$$

This order provides a clue to the appropriate scaling to find the second solution.

We define a new variable y of order unity by the scaling

$$y = \frac{x}{1/\epsilon} = \epsilon x.$$

The perturbed quadratic equation $\epsilon x^2 + 2x + 1 = 0$ becomes

$$\begin{aligned}&\epsilon(y/\epsilon)^2 + 2(y/\epsilon) + 1 = 0,\\ &y^2 + 2y + \epsilon = 0,\end{aligned}$$

where each term now has an order determined by its coefficient.

We now use a regular perturbation series

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$$

to approximate the large solution.

Substitution of the series into the quadratic equation gives

$$(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots)^2 + 2(y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots) + \epsilon = 0.$$

Equating the coefficients of like powers of ϵ gives

$$\begin{aligned}&y_0^2 + 2y_0 = 0,\\ &2y_0 y_1 + 2y_1 + 1 = 0, \text{ etc.}\end{aligned}$$

This gives

$$y = -\frac{1}{2} + \frac{\epsilon}{2} + \cdots.$$

Thus the second solution of the perturbed quadratic equation $\epsilon x^2 + 2x + 1 = 0$ is

$$x = \frac{y}{\epsilon} = -\frac{2}{\epsilon} - \frac{1}{2} + \cdots.$$

The two roots of the perturbed quadratic equation are of different orders, and one expansion does not give both.

The reasoning used in this example is called **dominant balancing**: careful examination of each term leads to which ones combine to give a dominant balance.

Example §3.2.2. We consider the perturbed boundary value problem

$$\begin{aligned} &\epsilon y'' + (1+\epsilon)y' + y = 0,\ 0 < x < 1,\ 0 < \epsilon \ll 1 \\ &y(0) = 0,\ y(1) = 1. \end{aligned}$$

Substitution of the regular perturbation series

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots$$

into the ODE gives

$$\begin{aligned} &\epsilon(y_0'' + \epsilon y_1'' + \epsilon^2 y_2'' + \cdots) + (y_0' + \epsilon y_1' + \epsilon^2 y_2' + \cdots) \\ &\quad + \epsilon(y_0' + \epsilon y_1' + \epsilon^2 y_2' + \cdots) + (y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots) = 0. \end{aligned}$$

Equating coefficients of like powers gives

$$\begin{aligned} &y_0' + y_0 = 0, \\ &y_0'' + y_1' + y_0' + y_1 = 0,\ \text{etc.} \end{aligned}$$

The boundary conditions force

$$\begin{aligned} &y_0(0) = 0,\ y_0(1) = 1, \\ &y_1(0) = 0,\ y_1(1) = 0,\ \text{etc.} \end{aligned}$$

The boundary value problem for y_0 is

$$y_0' + y_0 = 0,\ y_0(0) = 0,\ y_0(1) = 1.$$

The general solution of the first-order ODE is $y_0(x) = ce^{-x}$ for an arbitrary constant c.

Applying the boundary conditions leads to the linear system of equations

$$0 = c,\ 1 = ce^{-1}.$$

This system is inconsistent (no solution for c).

Regular perturbation has failed at the first step.

The first boundary condition $y_0(0) = 0$ forces $y_0(x) = 0$, while the second boundary condition forces $y_0(x) = e^{1-x}$.

Ignoring the term $\epsilon y''$ in the original ODE led to a first-order ODE which is very different from a second-order ODE.

Here again the term $\epsilon y''$ might be large for small ϵ and small x.