Math 521 Lecture #20
§3.2.3: Boundary Layers

We again consider the perturbed boundary value problem

$$\begin{aligned}&\epsilon y'' + (1+\epsilon)y' + y = 0,\ 0 < x < 1,\ 0 < \epsilon \ll 1\\ &y(0) = 0,\ y(1) = 1.\end{aligned}$$

Recall that substitution of the regular perturbation series

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots$$

into the BVP led to the first-order BVP

$$y_0' + y_0 = 0,\ y_0(0) = 0,\ y_0(1) = 1$$

which had no solution.

This first-order BVP has a different character than the original second-order BVP.

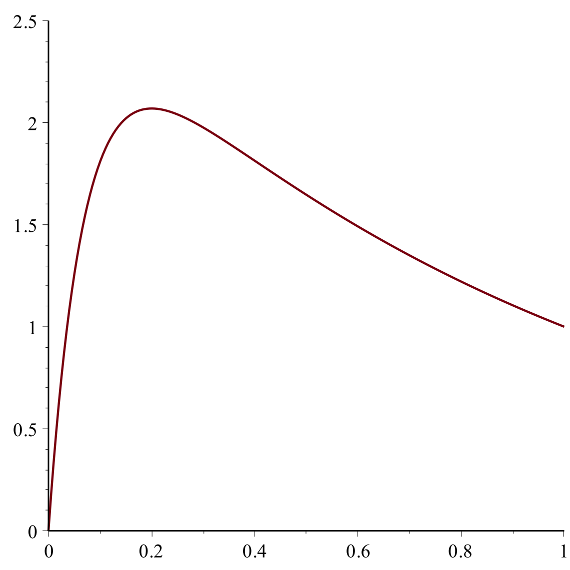
When this occurs, we should be suspicious of the regular perturbation method.

Since the second-order ODE is linear, we can explicitly solve the second-order BVP and find a modification to the regular perturbation method that will give a good approximation.

The explicit solution of the second-order BVP is

$$y(x) = \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}}.$$

Here is the graph of this solution for $\epsilon = 0.07$.



From this graph we see what the issue is: the solution $y(x)$ is changing very rapidly near $x = 0$, so the term

$$\epsilon\frac{d^2y}{dy^2}$$

in the second-order ODE, is not small when ϵ is small.

The small interval where $y(x)$ is experiencing this rapid change is called a **boundary layer**.

The solution $y(x)$ changes more slowly away from $x = 0$, on a much larger interval called the **outer layer**.

This difference in the behavior of y between the boundary layer and the outer layer indicates that two spatial scales in x are needed, one for each layer.

To determined an appropriate scaling in the outer layer, we investigate the terms in the ODE that involve ϵ, namely y'' and y'.

These derivative of the solution of the BVP are

$$y'(x) = \frac{-e^{-x} + \epsilon^{-1}e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}},$$
$$y''(x) = \frac{e^{-x} - \epsilon^{-2}e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}}.$$

For $x = O(1)$, i.e, far away from the boundary layer, say $x = 1/2$, we have

$$y'(1/2) = \frac{-e^{1/2} + \epsilon^{-1}e^{-1/2\epsilon}}{e^{-1} - e^{-1/\epsilon}}.$$

This is $O(1)$ because

$$\lim_{\epsilon\to 0}\left|\frac{-e^{1/2} + \epsilon^{-1}e^{-1/2\epsilon}}{e^{-1} - e^{-1/\epsilon}}\right| = \left|\frac{e^{-1/2}}{e^{-1}}\right|.$$

The second derivative value

$$y''(1/2) = \frac{e^{-1/2} - \epsilon^{-2}e^{-1/2\epsilon}}{e^{-1} - e^{-1/\epsilon}}$$

is $O(1)$ because

$$\lim_{\epsilon\to 0}\left|\frac{e^{-1/2} - \epsilon^{-2}e^{-1/2\epsilon}}{e^{-1} - e^{-1/\epsilon}}\right| = \left|\frac{\epsilon^{-1/2}}{e^{-1}}\right|.$$

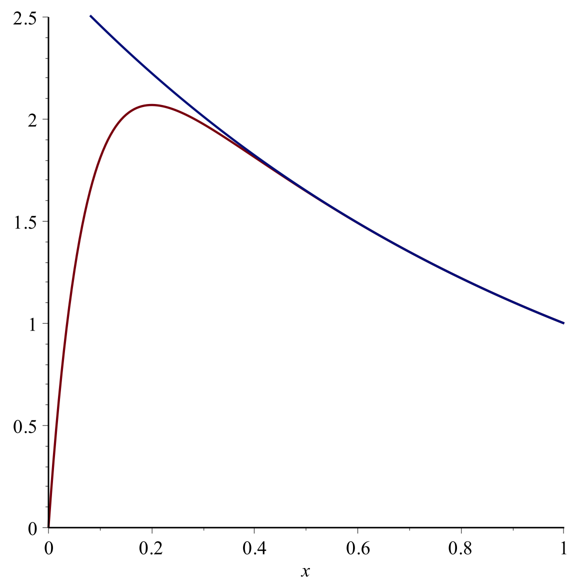
Thus the terms $\epsilon y''(x)$ and $\epsilon y'(x)$ are small in the outer layer and can be safely ignored for small ϵ along with the boundary condition $y(0) = 0$ that is not in the outer layer.

Thus no scaling of x is needed, and the IVP we obtain by setting $\epsilon = 0$ is

$$y' + y = 0,\ y(1) = 0,$$

whose solution $y_o(x) = e^{1-x}$ for $x = O(1)$ is an approximation of $y(x)$ in the outer layer, or an **outer approximation**.

Here is the graph of $y_o(x)$ with $y(x)$ for $\epsilon = 0.07$.



To determine the appropriate spatial scale for the boundary layer, we investigate the second derivative of the solution of the BVP:

$$y''(x) = \frac{e^{-x} - \epsilon^{-2} e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}}.$$

For ϵ small and x close to 0, say $x = \epsilon$, we have

$$y''(\epsilon) = \frac{e^{-\epsilon} - \epsilon^{-2} e^{-1}}{e^{-1} - e^{-1/\epsilon}}.$$

What is the order of $y''(\epsilon)$?

The presence of the ϵ^{-2} in the numerator of $y(\epsilon)$ suggest that $y(\epsilon) = O(\epsilon^{-2})$.

This is indeed the case because

$$\lim_{\epsilon \to 0} \left| \frac{\frac{e^{-\epsilon} - \epsilon^{-2} e^{-1}}{e^{-1} - e^{-1/\epsilon}}}{\epsilon^{-2}} \right| = \lim_{\epsilon \to 0} \left| \frac{\epsilon^2 e^{-\epsilon} - e^{-1}}{e^{-1} - e^{-1/\epsilon}} \right| = 1.$$

This says that y'' is very large in the boundary layer, so that $\epsilon y''$ is not small (as would be needed for the regular perturbation method to succeed).

In particular, we have $\epsilon y''(\epsilon) = O(\epsilon^{-1})$ because

$$\lim_{\epsilon \to 0} \left| \frac{\epsilon y''(\epsilon)}{\epsilon^{-1}} \right| = \lim_{\epsilon \to 0} \left| \frac{\epsilon e^{-\epsilon} - \epsilon^{-1} e^{-1}}{\epsilon^{-1}} \right| = \lim_{\epsilon \to 0} \left| \frac{\epsilon^2 e^{-\epsilon} - e^{-1}}{e^{-1} - e^{-1/\epsilon}} \right| = 1.$$

This says that the ϵ in $\epsilon y''$ does not reflect the magnitude of order of this term.

We will show in the next lecture that through dominant balancing, an appropriate scaling of x in the boundary layer is given by $\xi = x/\epsilon$.

For now, with ϵ close to 0, the term $e^{-1/\epsilon}$ in the solution $y(x)$ is close to 0, so that

$$y(x) \approx \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1}} = e^{1-x} - e^{1-x/\epsilon}.$$

For x in the boundary layer, i.e., close to 0 or $x = O(1/\epsilon)$, we have that

$$y(x) \approx e - e^{1-x/\epsilon}.$$

This approximate solution captures the rapid changes in y in the boundary layer, and is called the inner approximation and denoted by $y_i(x)$.

Here is a graph of $y(x)$, $y_o(x)$, and $y_i(x)$ for $\epsilon = 0.07$.