Math 521 Lecture #21
§3.3.1, 3.3.2: Inner, Outer Approximations; Matching

We illustrate the first steps of *boundary layer analysis* with the boundary value problem

$$\begin{aligned}
&\epsilon y'' + (1+\epsilon)y' + y = 0,\ 0 < x < 1,\ 0 < \epsilon \ll 1,\\
&y(0) = 0,\ y(1) = 1.
\end{aligned}$$

We have identified an outer layer $x = O(1)$ where $\epsilon y''$ and $\epsilon y'$ are small so that we can safely ignore them to get an outer approximation $y_o(x) = e^{1-x}$ determined by

$$y' + y = 0,\ y(1) = 1.$$

In the boundary layer, where x is close to 0, the values of $y(x)$, $y'(x)$, and $y''(x)$ are changing rapidly.

We seek for a time scale appropriate to this rapid behavior by setting

$$\xi = \frac{x}{\delta(\epsilon)},\ Y(\xi) = y(\delta(\xi)t)$$

for a yet to be determined function δ.

For this time scale the ODE becomes

$$\frac{\epsilon}{\delta(\epsilon)^2}Y''(\xi) + \frac{(1+\epsilon)}{\delta(\epsilon)}Y'(\xi) + Y(\epsilon) = 0.$$

The coefficients of the four terms in the ODE are

$$\frac{\epsilon}{\delta(\epsilon)^2},\ \frac{1}{\delta(\epsilon)},\ \frac{\epsilon}{\delta(\epsilon)},\ 1.$$

Each of these coefficients, for a correct scaling, will reflect the order of magnitude of corresponding term in the ODE.

To determine an appropriate choice of δ we use dominant balancing among pairings of the four coefficients, with a special focus on the coefficient of the second-derivative term that plays a significant role in the boundary layer.

There are three pairs to consider:

(i) the coefficients $\epsilon/\delta(\epsilon)^2$ and $1/\delta(\epsilon)$ are of the same order, while $\epsilon/\delta(\epsilon)$ and 1 are small by comparison,

(ii) the coefficients $\epsilon/\delta(\epsilon)^2$ and 1 are of the same order, while $1/\delta(\epsilon)$ and $\epsilon/\delta(\epsilon)$ are small by comparison,

(iii) the coefficients $\epsilon/\delta(\epsilon)^2$ and $\epsilon/\delta(\epsilon)$ are of the same order, while $1/\delta(\epsilon)$ and 1 are small by comparison.

We will use the symbol $\sim$ to denote "of the same order" in the following analysis.

In case (ii), $\epsilon/\delta(\epsilon)^2 \sim 1$ implies that $\delta(\epsilon)^2 \sim \epsilon$, and so $\delta(\epsilon) = O(\sqrt{\epsilon})$.

Then $1/\delta(\epsilon)$ is not small compared with 1, so this case is inconsistent,

In case (iii), $\epsilon/\delta(\epsilon)^2 \sim \epsilon/\delta(\epsilon)$ implies that $\delta(\epsilon) \sim 1$, or that $\delta(\epsilon) = O(1)$.

This implies that $\xi \sim x$ which leads to the outer layer and its approximation.

That leaves case (i), $\epsilon/\delta(\epsilon)^2 \sim 1/\delta(\epsilon)$ which implies that $\delta(\epsilon) = O(\epsilon)$.

It follows that both $\epsilon/\delta(\epsilon)^2$ and $1/\delta(\epsilon)$ are both of order $1/\epsilon$, which is large compared with $\epsilon/\delta(\epsilon) \sim 1$ and 1.

A consistent scaling is $\delta(\epsilon) = O(\epsilon)$, or

$$\delta(\epsilon) = \epsilon.$$

With the choice of scaling, the ODE becomes

$$\frac{1}{\epsilon}Y''(\xi) + \frac{(1+\epsilon)}{\epsilon}Y'(\xi) + Y(\xi) = 0.$$

Multiplying through by ϵ gives

$$Y''(\xi) + Y'(\xi) + \epsilon Y'(\xi) + \epsilon Y(\xi) = 0.$$

To this we can substitute a regular perturbation $Y(\xi) = Y_i(\xi) + O(\epsilon)$ which gives for the leading-order term Y_i the ODE

$$Y_i'' + Y_i' = 0,$$

which is the unperturbed equation.

The general solution of this second-order ODE is

$$Y_i(\xi) = C_1 + C_2 e^{-\xi}.$$

Because the boundary condition $y(0) = 0$, or $Y_i(0) = 0$, lies in the boundary layer, we apply it to this general solution to get

$$Y_i(\xi) = C_1(1 - e^{-\xi}).$$

Undoing the time scaling we have the inner approximation

$$y_i(x) = C_1(1 - e^{-x/\epsilon})$$

for $x = O(\epsilon)$.

We have obtained the inner and outer approximations

$$y_i(x) = C_1(1 - e^{-x/\epsilon}),\ x = O(\epsilon),$$
$$y_o(x) = e^{1-x},\ x = O(1).$$

It remains to determine the value of the arbitrary constant C_1 in the inner approximation.

The means by which the value of C is determined is called **matching**.

This does NOT mean that we find a value ϵ_0 at which we impose $y_i(\epsilon_0) = y_o(\epsilon_0)$ which would give a continuous patching together of the inner and outer approximations.

There really isn't on choice of ϵ_0 that works because the **width** of the boundary layer given by $\delta(\epsilon)$ goes to 0 as $\epsilon \to 0$.

Rather, matching is about constructing a single composite expansion in ϵ that is uniformly valid on $[0, 1]$ as $\epsilon \to 0$.

The idea of matching is to have the inner and outer approximations agree to some order on an interval overlapping the boundary layer and the outer layer.

This can be done by finding an intermediate scale, one that is between the boundary layer and the outer layer.

With $x = O(\epsilon)$ in the boundary layer and $x = O(1)$ in the outer layer, values of x for which $x = O(\sqrt{\epsilon})$ characterize the overlap domain because $\sqrt{\epsilon}$ is between ϵ and 1.

In the overlapping domain we use the scaled variable

$$\eta = \frac{x}{\sqrt{\epsilon}}.$$

The condition for the desired matching between the inner and outer approximations is given in terms of the intermediate variable η: for fixed η we require that

$$\lim_{\epsilon \to 0^+} y_o(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0^+} y_i(\sqrt{\epsilon}\eta).$$

For the outer approximation $y_o(x) = e^{1-x}$, we have

$$\lim_{\epsilon \to 0^+} y_o(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0^+} e^{1-\sqrt{\epsilon}\eta} = e.$$

On the other hand, for the inner approximation we have

$$\lim_{\epsilon \to 0^+} y_i(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0^+} C_1(1 - e^{-\eta/\sqrt{\epsilon}}) = C_1.$$

The matching condition then requires that $C_1 = e$.

We can avoid the intermediate variable when finding the leading order term, because the matching condition is the simply

$$e = \lim_{x \to 0^+} y_o(x) = \lim_{\xi \to \infty} Y_i(\xi) = C_1.$$

This says that the limit of the outer approximation as the outer variable moves into the boundary layer *must* equal the limit of the inner approximation as the inner variable moves into the outer layer.

Note that for fixed η, as $\epsilon \to 0^+$ we have $x = \eta\sqrt{\epsilon} = O(\sqrt{\epsilon})$, so that $\xi = x/\epsilon = O(1/\sqrt{\epsilon})$, so that $\xi \to \infty$.