Math 521 Lecture #22
§3.3.3: Uniform Approximations

Recall for the boundary value problem,

$$\epsilon y'' + (1+\epsilon)y' + y = 0,\ 0 < x < 1,\ 0 < \epsilon \ll 1,$$
$$y(0) = 0,\ y(1) = 1,$$

that we obtained in the boundary layer near $x = 0$ the inner approximation

$$y_i(x) = e(1 - e^{-x/\epsilon}) = e - e^{1-x/\epsilon},\ x = O(\epsilon),$$

and in the outer layer the outer approximation

$$y_o(x) = e^{1-x},\ x = O(1).$$

We show how to combine these approximations into one uniform approximation on $[0, 1]$ of the exact solution

$$y(x,\epsilon) = \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}} \approx \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1}} = e^{1-x} - e^{1-x/\epsilon}.$$

In the overlap domain, both the inner and outer approximations are roughly equal to e, the common limit of the matching condition,

$$\lim_{\epsilon\to 0^+} y_0(\sqrt{\epsilon}\eta) = \lim_{\epsilon\to 0^+} y_i(\sqrt{\epsilon}\eta)$$

for fixed value of the intermediate variable $\eta = x/\sqrt{\epsilon}$.

Naively adding the inner and outer approximations together gives

$$y_i(x) + y_o(x) = e^{1-x} + e - e^{1-x/\epsilon}$$

which implies that approximately

$$2e - e^{1-x/\epsilon} \text{ for } x = O(\epsilon) \text{ and } e^{1-x} + e \text{ for } x = O(1).$$

Each of these is off by the common limit of the matching condition.

To fix this, we subtract the common limit of the matching condition from the sum of the inner and outs approximations to obtain

$$y_u(x,\epsilon) = y_i(x) + y_0(x) - e = e^{1-x} - e^{1-x/\epsilon}.$$

This approximation is actually a solution of the ODE because

$$\begin{aligned}
&\epsilon y_u'' + (1+\epsilon)y_u' + y_u \\
&\quad = \epsilon(e^{1-x} - \epsilon^{-2}e^{1-x/\epsilon}) + (1+\epsilon)(-e^{1-x} + \epsilon^{-1}e^{1-x/\epsilon}) + e^{1-x} - e^{1-x/\epsilon} \\
&\quad = 0.
\end{aligned}$$

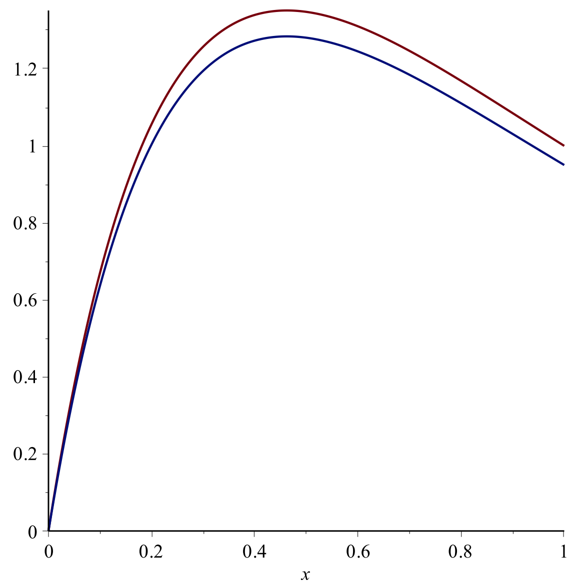
The approximation $y_u(x,\epsilon)$ satisfies the boundary condition $y(0) = 0$ exactly, while for the other boundary condition we have

$$y_u(1,\epsilon) = 1 - e^{1-1/\epsilon} \approx 1.$$

Is $y_u(x,\epsilon)$ a uniformly valid asymptotic approximation of $y(x,\epsilon)$ on $I = [0,1]$ as $\epsilon \to 0^+$?

It is because (as shown in the Appendix) we have $E(x,\epsilon) = O(e^{-1/\epsilon})$ uniformly on I as $\epsilon \to 0^+$.

Here are the graphs of the exact solution $y(x,\epsilon)$ (red or upper curve) and the uniform approximation $y_u(x,\epsilon)$ (blue or lower curve) for $\epsilon = 0.25$.



We used the not so small value of $\epsilon = 0.25$ to see the difference between the exact and approximate solutions because the rate of convergence has order $e^{-1/\epsilon}$ which is super fast (faster than any polynomial in ϵ, see Appendix).

Using $\epsilon = 0.07$ we cannot see the difference between the exact and the approximate solutions at the scale of the graphs.

Example. Use singular perturbation methods to find a uniformly valid approximation to the problem

$$\begin{aligned}&\epsilon y'' + y' = 2x,\ 0 < x < 1,\ 0 < \epsilon \ll 1,\\ &y(0) = 1,\ y(1) = 1.\end{aligned}$$

The general solution of the unperturbed ODE $y' = 2x$ is $y = x^2 + C$ which cannot satisfy both boundary conditions.

Consequently we assume a boundary layer at $x = 0$ and an outer layer that includes $x = 1$.

The boundary condition $y(1) = 1$ belongs to the outer layer, which implies that an outer approximation is

$$y_o(x) = x^2,\ x = O(1).$$

We assume that the width of the boundary layer has the form $\delta(\epsilon)$ and rescale near $x = 0$ by

$$\xi = \frac{x}{\delta(\epsilon)}, \quad Y(\xi) = y(\delta(\epsilon)\xi).$$

With this scaling the ODE becomes

$$\frac{\epsilon}{\delta(\epsilon)^2} Y''(\xi) + \frac{1}{\delta(\epsilon)} Y'(\xi) = 2\delta(\epsilon)\xi.$$

We look for a dominant balance that reflects the order of magnitudes of the terms in the ODE.

If $\epsilon/\delta(\epsilon)^2 \sim 2\delta(\epsilon)$ is the dominant balance, then $\delta(\epsilon) = O(\epsilon^{1/3})$.

This implies that the coefficient $1/\delta(\epsilon) = O(e^{-1/3})$ is not small when compared with the assumed dominant coefficients.

If instead, $\epsilon/\delta(\epsilon)^2 \sim 1/\delta(\epsilon)$ is the dominant balance, then $\delta(\epsilon) = O(\epsilon)$.

This implies that the coefficient $2\delta(\epsilon) = O(\epsilon)$ which is small when compared with dominant coefficients $\epsilon/\delta(\epsilon)$ and $1/\delta(\epsilon)$ each of which are $O(\epsilon^{-1})$.

We therefore choose $\delta(\epsilon) = \epsilon$ which gives a consistent scaling of the ODE, i.e., the coefficient reflect the order of magnitude of the corresponding terms.

The ODE becomes

$$Y''(\xi) + Y'(\xi) = 2\epsilon^2 \xi$$

so the inner approximation Y_i satisfies

$$Y_i'' + Y_i' = 0.$$

Solving this gives

$$Y_i(\xi) = C_1 + C_2 e^{-\xi},$$

or in terms of the variables x and y,

$$y_i(x) = C_1 + C_2 e^{-x/\epsilon}.$$

The boundary condition $y(0) = 1$ implies that $C_1 = 1 - C_2$, so that

$$y_o(x) = (1 - C_2) + C_2 e^{-x/\epsilon}.$$

In an overlap domain of order $\sqrt{\epsilon}$ and an appropriate intermediate scaled variable $\eta = x/\sqrt{\epsilon}$, the matching condition (that determines the value of C_2) for fixed η is

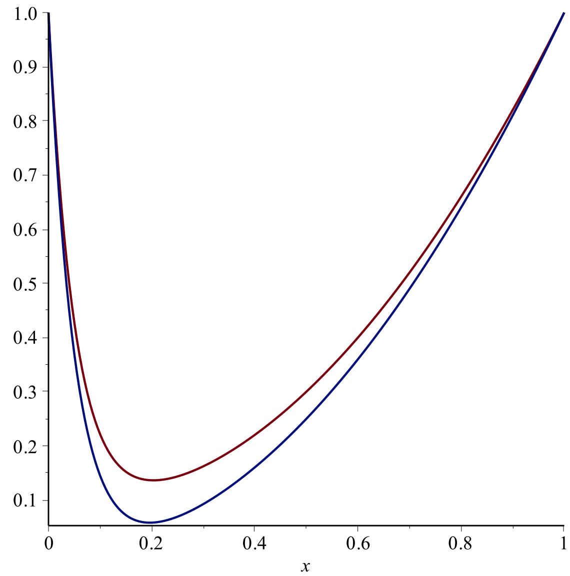
$$0 = \lim_{\epsilon \to 0^+} \epsilon\eta^2 = \lim_{\epsilon \to 0^+} y_0(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0^+} y_i(\sqrt{\epsilon}\eta) = \lim_{\epsilon \to 0^+} \left[(1 - C_2) + C_2 e^{-\eta/\sqrt{\epsilon}}\right] = 1 - C_2.$$

This gives $C_2 = 1$ and an inner approximation of $y_i(x) = e^{-x/\epsilon}$, $x = O(\epsilon)$.

A uniform composite approximation is inner plus outer minus the common limit in the overlap domain:

$$y_u(x) = x^2 + e^{-x/\epsilon} - 0 = x^2 + e^{-x/\epsilon}.$$

Here are the graphs of the exact solution (the red or upper curve) and the uniformly valid approximation $y_u(x)$ (the blue or lower curve) when $\epsilon = 0.05$.



Appendix.

Calculation of the rate of uniform convergence of the approximation as $\epsilon \to 0^+$.

The error function for the approximation $y_u(x,\epsilon)$ is

$$\begin{aligned}
E(x,\epsilon) &= y(x,\epsilon) - y_u(x,\epsilon)\\
&= \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}} - \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1}}\\
&= \left(e^{-x} - e^{-x/\epsilon}\right)\left(\frac{1}{e^{-1} - e^{-1/\epsilon}} - \frac{1}{e^{-1}}\right)\\
&= \left(e^{-x} - e^{-x/\epsilon}\right)\left(\frac{e^{-1/\epsilon}}{(e^{-1} - e^{-1/\epsilon})e^{-1}}\right)\\
&= y(x,\epsilon)e^{1-1/\epsilon}.
\end{aligned}$$

For each $0 < \epsilon \ll 1$, the continuous nonnegative function $y(x,\epsilon)$ has one critical point in the compact $[0,1]$ at which is achieves its maximum value of

$$y_{\max}(\epsilon) = y\left(-\frac{\ln\epsilon}{\frac{1}{\epsilon} - 1}, \epsilon\right) = \frac{1}{e^{-1} - e^{-1/\epsilon}}\left(\exp\left(\frac{\ln\epsilon}{\frac{1}{\epsilon} - 1}\right) - \exp\left(\frac{\ln\epsilon}{1-\epsilon}\right)\right).$$

This implies that for each $0 < \epsilon \ll 1$ that $y(x,\epsilon) \le y_{\max}(\epsilon)$ for all $x \in [0,1]$.

Since

$$\lim_{\epsilon\to 0^+} \frac{\ln\epsilon}{\frac{1}{\epsilon} - 1} = \lim_{\epsilon\to 0^+} \frac{1/\epsilon}{-1/\epsilon^2} = 0,\ \lim_{\epsilon\to 0^+} \frac{\ln\epsilon}{1-\epsilon} = -\infty,$$

we have that

$$\lim_{\epsilon\to 0^+} y_{\max}(\epsilon) = e.$$

For sufficiently small ϵ we then have that $y(x,\epsilon) \le 2e$ for all $x \in [0,1]$.

Thus we obtain

$$|E(x,\epsilon)| \le (2e)e^{1-1/\epsilon} = (2e^2)e^{-1/\epsilon}$$

which implies uniformly on $[0,1]$ that

$$E(x,\epsilon) = O(e^{-1/\epsilon}) \text{ as } \epsilon \to 0^+.$$

Thus $y_u(x,\epsilon)$ is a uniformly valid asymptotic approximation of $y(x,\epsilon)$ on $[0,1]$ as $\epsilon \to 0^+$.

To understand what this rate of convergence is, we compare with $O(\epsilon^n)$.

For any positive integer n we have (with $\alpha = 1/\epsilon$) that

$$\lim_{\epsilon\to 0^+} \frac{e^{-1/\epsilon}}{\epsilon^n} = \lim_{\epsilon\to 0^+} \frac{e^{-1/\epsilon}}{\epsilon^n} = \lim_{\alpha\to\infty} \alpha^n e^{-\alpha} = 0.$$

This implies for each $n \in \mathbb{N}$ that $E(x,\epsilon) = o(\epsilon^n)$ as $\epsilon \to 0^+$, so that $E(x,\epsilon) = O(\epsilon^n)$ as $\epsilon \to 0^+$.