Math 521 Lecture #23
§3.3.4: General Procedures

The single boundary layers we have explored so far have occurred the left endpoint of the interval.

Boundary layers may occur at other points in an interval, with even several boundary layers in one interval.

When the boundary layer occurs at the right endpoint of the interval, the boundary layer analysis is identical, except for two things.

If b is the right endpoint of the interval, the scale transformation to define the inner variable is the different

$$\xi = \frac{b-x}{\delta(\epsilon)}.$$

The function $Y(\xi)$ is accordingly defined by $Y(\xi) = y(b - \delta(\epsilon)\xi)$.

This implies that the first derivatives have a different relationship,

$$\frac{dy}{dt} = -\frac{1}{\delta(\epsilon)}\frac{dY}{d\xi},$$

while the second derivatives have the same relationship,

$$\frac{d^2y}{dt^2} = \frac{1}{\delta(\epsilon)^2}\frac{d^2Y}{d\xi^2}.$$

The matching condition at a right boundary layer is

$$\lim_{\xi\to\infty} Y_i(\xi) = \lim_{x\to b^-} y_o(x).$$

The singular perturbation method is not universal: for some classes of differential equations it works well, while for other classes it the needed modifications can be significant.

We present one class of second-order BVPs to which the singular perturbation method applies, for which the boundary layer can be completely characterized, thus enabling a uniformly valid asymptotic approximation.

Theorem 3.12. Suppose $p(x)$ and $q(x)$ are continuous on $[0,1]$ with $p(x) > 0$. For the boundary value problem

$$\begin{aligned}&\epsilon y'' + p(x)y' + q(x)y = 0,\ 0 < x < 1,\ 0 < \epsilon \ll 1,\\ &y(0) = a,\ y(1) = b,\end{aligned}$$

there exists a boundary layer at $x = 0$ with inner approximation

$$y_i(x) = C_1 + (a - C_1)\exp\left(-\frac{p(0)x}{\epsilon}\right)$$

and outer approximation

$$y_o(x) = b\exp\left(\int_x^1 \frac{q(s)}{p(s)}ds\right),$$

where

$$C_1 = b \exp\left(\int_0^1 \frac{q(s)}{p(s)} dx\right).$$

The function

$$y_u(x) = y_i(x) + y_o(x) - C_1 = (C_1 - a)\exp\left(-\frac{p(0)x}{\epsilon}\right) + b\exp\left(\int_x^1 \frac{q(s)}{p(s)} ds\right)$$

is a uniformly valid asymptotic expansion, where, for $y(x)$ the exact solution, we have $y(x) - y_u(x) = O(\epsilon)$ uniformly on $[0, 1]$ as $\epsilon \to 0$.

Proof. We show that the assumption of a boundary layer at $x = 0$ is consistent and leads to the outer and inner approximations above.

If the boundary layer is at $x = 0$, then the outer approximation $y_o(x)$ will satisfy

$$p(x)y_o' + q(x)y_o = 0, \ y(1) = b.$$

Separating variables gives

$$\frac{dy_o}{y_o} = -\frac{q(x)}{p(x)} dx.$$

Integrating over the interval $[x, 1]$ and using $y_o(1) = b$ and $p(x) > 0$ gives

$$\ln|b| - \ln|y_0(x)| = -\int_x^1 \frac{q(s)}{p(s)} ds.$$

Solving for $y_o(x)$ gives

$$y_0(x) = b\exp\left(\int_x^1 \frac{q(s)}{p(s)} ds\right).$$

In the boundary layer, we introduce the scaled variable

$$\xi = \frac{x}{\delta(\epsilon)}$$

for a yet-to-be determined function $\delta(\epsilon)$ with the property that $\delta(\epsilon) \to 0$ as $\epsilon \to 0^+$.

Setting $Y(\xi) = y(\delta(\epsilon)\xi)$, the ODE in Y is

$$\frac{\epsilon}{\delta(\epsilon)^2} Y'' + \frac{p(\delta(\epsilon)\xi)}{\delta(\epsilon)} Y' + q(\delta(\epsilon)\xi)Y = 0.$$

As $\epsilon \to 0^+$, the coefficients in this ODE behave like

$$\frac{\epsilon}{\delta(\epsilon)^2}, \ \frac{p(0)}{\delta(\epsilon)}, \ q(0)$$

by the continuity of p and q at $x = 0$.

The dominant balance here is $\epsilon/\delta(\epsilon)^2 \sim p(0)/\delta(\epsilon)$, which implies that $\delta(\epsilon) = O(\epsilon)$ as $\epsilon \to 0^+$.

We can therefore choose $\delta(\epsilon) = \epsilon$.

The ODE the becomes

$$Y'' + p(\epsilon\xi)Y' + \epsilon q(\epsilon\xi)Y = 0.$$

The inner approximation Y_i then satisfies

$$Y_i'' + p(0)Y_i' = 0.$$

The general solution of this is

$$Y_i(\xi) = C_1 + C_2 e^{-p(0)\xi}.$$

The boundary condition $y(0) = a$ becomes $Y_i(0) = a$, so that

$$a = C_1 + C_2.$$

Thus the inner approximation is

$$Y_i(\xi) = C_1 + (a - C_1)e^{-p(0)\xi}.$$

Back in original variables, the inner approximation is

$$y_i(x) = C_1 + (a - C_1)\exp\left(-\frac{p(0)x}{\epsilon}\right).$$

For the matching condition, we use the intermediate variable $\eta = x/\sqrt{\epsilon}$ in an overlap domain.

For fixed η, the matching condition is

$$\lim_{\epsilon\to 0^+} y_i(\sqrt{\epsilon}\eta) = \lim_{\epsilon\to 0^+} y_o(\sqrt{\epsilon}\eta).$$

Filling this in gives

$$\lim_{\epsilon\to 0^+}\left[C_1 + (a - C_1)\exp\left(-\frac{p(0)\eta}{\sqrt{\epsilon}}\right)\right] = \lim_{\epsilon\to 0^+} b\exp\left(\int_{\sqrt{\epsilon}\eta}^1 \frac{q(s)}{p(s)}ds\right).$$

This forces

$$C_1 = b\exp\left(\int_0^1 \frac{q(s)}{p(s)}ds\right).$$

The proof that the approximation

$$y_u(x) = y_i(x) + y_o(x) - C_1 = (a - C_1)\exp\left(-\frac{p(0)x}{\epsilon}\right) + b\exp\left(\int_x^1 \frac{q(s)}{p(s)}ds\right)$$

satisfies $y(x) - y_u(x) = O(\epsilon)$ is beyond this course. $\square$