Math 521 Lecture #24
§3.4.1: Damped Mass-Spring Systems

Initial value problems with a small parameter may also have boundary layers, which are called **initial layers**.

We will illustrate an initial layer for a damped mass-spring system.

Let y be the displacement of the object of mass m from its equilibrium position.

We assume that the forces acting on the object are

$$F_{\text{spring}} = -ky,\ F_{\text{damping}} = -a\dot{y},$$

for positive constants k and a.

The model for the position of the object in the mass-spring system is

$$m\ddot{y} + a\dot{y} + ky = 0,\ t > 0.$$

Assuming the initial displacement is 0, we put the object into motion by imparting a positive impulse I.

This means that the initial conditions are

$$y(0) = 0,\ m\dot{y}(0) = I.$$

What happens when m is very small?

To investigate the initial layer at $t = 0$, we put the IVP into dimensionless form.

The dimensions of the four parameters in the IVP are

$$[m] = M,\ [a] = MT^{-1},\ [k] = MT^{-2},\ [I] = MLT^{-1}.$$

From these there are three time scales,

$$\frac{m}{a},\ \sqrt{\frac{m}{k}},\ \frac{a}{k}$$

which corresponding to balancing inertia (the mass) and the damping, the inertia and the spring, and the damping and the spring.

Three possible displacement scales are

$$\frac{I}{a},\ \frac{I}{\sqrt{km}},\ \frac{aI}{km}.$$

Because we are assuming $m \ll 1$, we have that $m/a \ll 1$, $\sqrt{m/k} \ll 1$, $I/\sqrt{km} \gg 1$, and $aI/km \gg 1$.

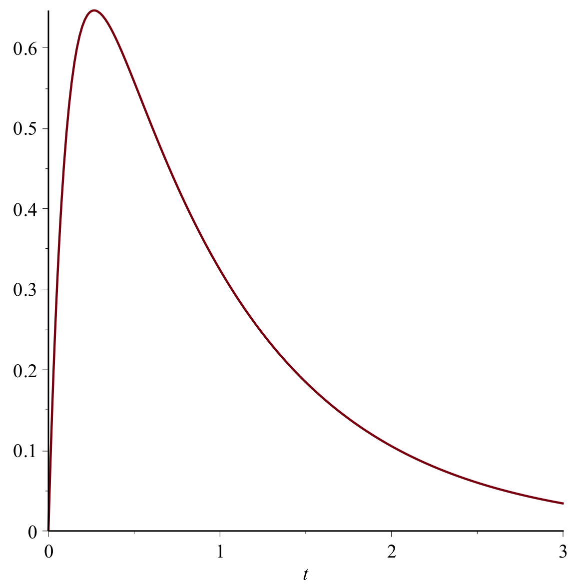
Such relations are needed when determining appropriate scales for the independent and dependent variables, because the scales should be chosen so that these variables in scaled form both have order 1.

Before we make the choice of scales, we first analyze the motion of the object.

Because m is small the impulse I will impart a large initial velocity, so there is a rapid change in the displacement to a maximum value, at which point the spring will cause the displacement to return to 0.

With m small, the mass-spring system is overdamped, i.e., $a^2 - 4km > 0$, and there will be no oscillations in the displacement.

Here is the graph of the solution for $m = 0.1$, $a = 1$, $k = 1$, and $I = 0.1$.



The quick rise followed by a gentle decay suggests two time scales are needed.

Near $t = 0$ we have what is called an **initial layer**, where the value of y is rapidly changing and a short time scale is needed.

In the outer layer, a time scale of order 1 seems appropriate.

The IVP has the characteristics of a singular perturbation problem: multiple time scales and a small quantity multiplying the highest order derivative.

Looking at the inner layer, either of the time scales m/a or $\sqrt{m/k}$ may be suitable since m is small.

The choice of $\sqrt{m/k}$ is poor because for small y the spring force ky is not a dominant term.

A good choose would be m/a because the damping force ay' is dominant term in initial layer where y' is large.

The term my'' is also dominant in the initial layer.

Looking at the possible displacement scales for the outer layer, the only one that seems suitable is I/a because the other two are large in comparison with the maximum displacement.

Looking at the possible time scales for the outer layer, only the third one a/k is suitable because it is of first order, while the other two are of smaller order.

For the outer layer the scalings we use to non dimensionless the IVP are

$$\bar{t} = \frac{t}{a/k} = \frac{kt}{a}, \ \bar{y} = \frac{y}{I/a} = \frac{ay}{I}.$$

The scaled variable $\bar{t}$ and $\bar{y}$ are both of order 1.

In the scaled variables, the IVP becomes

$$\begin{aligned} &\epsilon\bar{y}'' + \bar{y}' + \bar{y} = 0, \\ &\bar{y}(0) = 0, \ \epsilon\bar{y}'(0) = 1, \end{aligned}$$

where the prime means differentiation with respect to $\bar{t}$, and $\epsilon = mk/a^2 \ll 1$.

With this scaling the small term $\epsilon\bar{y}''$ occurs where we expect it, in the inertia term.

By singular perturbation, the leading order approximation in the outer layer is obtained by setting $\epsilon = 0$, i.e.,

$$\bar{y}' + \bar{y} = 0.$$

We do not include either initial condition because they lie in the initial layer.

The outer approximation is $\bar{y}_0(x) = Ce^{-\bar{t}}$.

To obtain an inner approximation we rescale according to

$$\tau = \frac{\bar{t}}{\delta(\epsilon)}, \ Y(\tau) = \bar{y}(\delta(\epsilon)\tau).$$

Then the scaled ODE becomes

$$\frac{\epsilon}{\delta(\epsilon)^2}Y''(\tau) + \frac{1}{\delta(\epsilon)}Y'(\tau) + Y(\tau) = 0.$$

The dominant balance is $\epsilon/\delta(\epsilon)^2 \sim 1/\delta(\epsilon)$ which gives the width of the initial layer as $\delta(\epsilon) = O(\epsilon)$.

The other balance $\epsilon/\delta(\epsilon)^2 \sim 1$ implies $\delta(\epsilon) = O(\sqrt{\epsilon})$ which implies the inconsistency of $1/\delta(\epsilon)$ not being small.

Setting $\delta(\epsilon) = \epsilon$ the ODE becomes

$$Y''(\tau) + Y'(\tau) + \epsilon Y(\tau) = 0.$$

The leading order approximation gives the inner approximation of

$$Y(\tau) = A + Be^{-\tau}.$$

Returning to $\bar{t}$ and $\bar{y}$ variables, we have

$$\bar{y}(\bar{t}) = A + Be^{-\bar{t}/\epsilon}.$$

The initial condition $y(0) = 0$ implies that $\bar{y}(0) = 0$, so that $B = -A$.

The other initial condition $\epsilon\bar{y}'(0) = 1$ implies that $B = -1$.

The inner approximation is $\bar{y}_i(\bar{t}) = 1 - e^{-\bar{t}/\epsilon}$.

We use matching to determine the value of the arbitrary constant C in the outer approximation.

We could pass to an intermediate scale $\eta = \bar{t}/\sqrt{\epsilon}$ to express the matching condition.

Instead we use simpler equivalent matching condition:

$$C = \lim_{\bar{t}\to 0^+} \bar{y}_0(\bar{t}) = \lim_{\tau\to\infty} Y(\tau) = A = 1.$$

Then a uniformly valid approximation is

$$\bar{y}_u(\bar{t}) = y_o(x) + y_i(x) - \text{common limit } = e^{-\bar{t}} - e^{-\bar{t}/\epsilon}.$$

Returning to the original variables we obtain the uniformly valid approximation

$$y_u(t) = \frac{I(e^{-kt/a} - e^{-at/m})}{a}.$$

For t in the initial layer we have

$$y_u(t) \approx \frac{I(1 - e^{-at/m})}{a},$$

which describes a rapidly increasing function with a time scale of m/a, in agreement with the time scale we thought would be appropriate in the initial layer.

For t away from 0, in the outer layer, we have

$$y_u(t) \approx \frac{Ie^{-kt/a}}{a},$$

which describes a slowing decaying function on a time scale of a/k, in agreement with the time scale we though would be appropriate in the outer layer.