Math 521 Lecture #26
§3.5: The WKB Approximation, Part II

Recall singularly perturbed equation $\epsilon^2 y'' + q(x)y = 0$ we obtained from the time-independent Schrödinger equation where $q(x) = E - V(x)$.

There are two cases to consider: the classical or oscillatory case when $q(x) > 0$ on some interval, and the non-classical or exponential case when $q(x) < 0$ on some interval.

The Nonoscillatory Case. To be explicit about $q(x) < 0$ for x on some interval I, we write $q(x) = -k(x)^2$ where $k(x) > 0$, so that the equation becomes

$$\epsilon^2 y'' - k(x)^2 y = 0.$$

If $k(x)$ were a constant k_0, then the solutions of $\epsilon y'' - k_0 y = 0$ would be real, rapidly increasing or decreasing exponential solutions of the form

$$\exp\left(\pm\frac{k_0 x}{\epsilon}\right).$$

This suggests making the substitution

$$y = \exp\left(\frac{u(x)}{\epsilon}\right)$$

in the ODE to get an ODE in $u(x)$.

Since

$$y'(x) = \frac{u'(x)}{\epsilon}\exp\left(\frac{u(x)}{\epsilon}\right),$$
$$y''(x) = \frac{u''(x)}{\epsilon} + \left(\frac{u'(x)}{\epsilon}\right)^2 \exp\left(\frac{u(x)}{\epsilon}\right),$$

the ODE becomes

$$\epsilon u''(x)\exp\left(\frac{u(x)}{\epsilon}\right) + (u'(x))^2\exp\left(\frac{u(x)}{\epsilon}\right) - k(x)^2\exp\left(\frac{u(x)}{\epsilon}\right) = 0.$$

Elimination the common exponential and setting $v = u'$ gives

$$\epsilon v' + v^2 - k(x)^2 = 0.$$

We substitute into the ODE the regular perturbation series

$$v(x) = v_0(x) + \epsilon v_1(x) + O(\epsilon^2)$$

to get for the leading-order, or $O(1)$, term and first corrective, or $O(\epsilon)$, term,

$$v_0^2 - k(x)^2 = 0, \;\; v_0' + 2v_0 v_1 = 0.$$

Solving these gives

$$v_0(x) = \pm k(x), \quad v_1 = -\frac{k'(x)}{2k(x)}.$$

Thus we have the asymptotical expansion

$$v(x) = \pm k(x) - \epsilon \frac{k'(x)}{2k(x)} + O(\epsilon^2).$$

Since $v' = u$, integrating of the asymptotic expansion for $v(x)$ gives an asymptotic expansion

$$u(x) = \pm \int_a^x k(\xi)\ d\xi - \frac{\epsilon}{2} \ln k(x) + O(\epsilon^2),$$

where a is an arbitrary constant.

Returning to the original dependent variable $y = \exp(u(x)/\epsilon)$ we have an asymptotic expansion

$$\begin{aligned} y(x) &= \exp\left(\pm\frac{1}{\epsilon}\int_a^x k(\xi)\ d\xi - \frac{\ln k(x)}{2} + O(\epsilon)\right) \\ &= \frac{1}{\sqrt{k(x)}} \exp\left(\pm\frac{1}{\epsilon}\int_a^x k(\xi)\ d\xi\right) + O(\epsilon). \end{aligned}$$

We have here two linearly independent approximations (one with the choice of $+$ in $\pm$, the other with the choice of $-$).

The arbitrary linear combination of these gives the WKB approximation in the nonoscillatory case:

$$y_{WKB}(x) = \frac{c_1}{\sqrt{k(x)}} \exp\left(\frac{1}{\epsilon}\int_a^x k(\xi)\ d\xi\right) + \frac{c_2}{\sqrt{k(x)}} \exp\left(-\frac{1}{\epsilon}\int_a^x k(\xi)\ d\xi\right).$$

Example 3.14. Find the WKB approximation for

$$\epsilon^2 y'' - (1+x)^2 y = 0, \ x > 0.$$

Here we have $k(x) = 1 + x$, so that

$$\int_0^x (1+\xi) d\xi = x + \frac{x^2}{2}.$$

The WKB approximation is

$$y_{WKB}(x) = \frac{c_1}{\sqrt{1+x}} \exp\left(\frac{x + x^2/2}{\epsilon}\right) + \frac{c_2}{\sqrt{1+x}} \exp\left(-\frac{x + x^2/2}{\epsilon}\right).$$

Equivalently we can write

$$y_{WKB}(x) = \frac{c_1}{\sqrt{1+x}} \cosh\left(\frac{x + x^2/2}{\epsilon}\right) + \frac{c_2}{\sqrt{1+x}} \sinh\left(-\frac{x + x^2/2}{\epsilon}\right).$$

The Oscillatory Case. Now we assume that $q(x) > 0$ on some interval I, and explicitly write $q(x) = k(x)^2$ for $k(x) > 0$.

Similar to non-oscillatory case, the substitution $y = \exp(iu(x)/\epsilon)$ leads to a second-order equation in u to which we apply the regular perturbation method.

This gives the WKB approximation in the oscillatory case to be

$$y_{WKB}(x) = \frac{c_1}{\sqrt{k(x}} \exp\left(\frac{i}{\epsilon}\int_a^x k(\xi)\ d\xi\right) + \frac{c_2}{\sqrt{k(x)}} \exp\left(-\frac{i}{\epsilon}\int_a^x k(\xi)\ d\xi\right).$$

This can be written equivalently as

$$y_{WKB}(x) = \frac{c_1}{\sqrt{k(x}} \cos\left(\frac{1}{\epsilon}\int_a^x k(\xi)\ d\xi\right) + \frac{c_2}{\sqrt{k(x)}} \sin\left(-\frac{1}{\epsilon}\int_a^x k(\xi)\ d\xi\right).$$

Example 3.15. For the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}y'' + \big(V(x) - E\big)y = 0$$

in the classical or oscillatory setting of $E - V(x) > 0$, we have (by a trig identity the combines the linear combination of cos and sin into one cos) the WKB approximation

$$y_{WKB}(x) = \frac{A}{(E - V(x))^{1/4}} \cos\left(\frac{\sqrt{2m}}{\hbar}\int_a^x \sqrt{E - V(\xi)}\ d\xi + \phi\right)$$

where A is the amplitude and ϕ is the phase.

Another application of the WKB approximation is the determination of large "eigenvalues" of simple differential equations.

Example 3.16. For a function $q(x) > 0$, consider the problem of solving the BVP

$$\begin{aligned} &y'' + \lambda q(x)y = 0,\ 0 < x < \pi, \\ &y(0) = 0,\ y(\pi) = 0, \end{aligned}$$

when λ is large, i.e., $\lambda \gg 1$.

A number λ is an **eigenvalue** of the BVP is there exists a nonzero (or nontrivial) solution of the BVP for that particular value of λ, and the nontrivial solution is called an **eigenfunction** corresponding to that particular value of λ.

By setting $\epsilon = 1/\sqrt{\lambda}$ and $k(x) = \sqrt{q(x)}$, the ODE becomes

$$\epsilon^2 y'' + k(x)^2 y = 0$$

to which we apply the WKB method when ϵ is small (or λ is big) to get

$$y_{WKB}(x) = \frac{1}{q(x)^{1/4}}\left[c_1 \cos\left(\sqrt{\lambda}\int_0^x \sqrt{q(\xi)}\ d\xi\right) + c_2 \sin\left(\sqrt{\lambda}\int_0^x \sqrt{q(\xi)}\ d\xi\right)\right].$$

The boundary condition $y(0) = 0$ forces $c_1 = 0$.

Then the boundary condition $y(\pi) = 0$ forces

$$0 = c_2 \sin\left(\sqrt{\lambda}\int_0^\pi \sqrt{q(\xi)}\; d\xi\right).$$

If $c_2 = 0$, then we would get the zero (or trivial) solution $y_{WKB}(x) = 0$.

To get $c_2 \neq 0$, we must have that

$$\sin\left(\sqrt{\lambda}\int_0^\pi \sqrt{q(\xi)}\; d\xi\right) = 0.$$

This requires that

$$\sqrt{\lambda}\sin_0^\pi \sqrt{q(\xi)}\; d\xi = n\pi$$

for a large positive integer n (to keep in line with large values of λ).

The large eigenvalues of the BVP are given approximately by

$$\lambda_n = n^2\pi^2\left(\int_0^\pi \sqrt{q(\xi)}\; d\xi\right)^{-2}.$$

Approximations of the corresponding eigenfunctions are given by taking $c_2 = 1$, i.e.,

$$y_n(x) = \frac{1}{q(x)^{1/4}}\sin\left(\frac{n\pi\int_0^x \sqrt{q(\xi)}\; d\xi}{\int_0^\pi \sqrt{q(\xi)}\; d\xi}\right).$$