Math 521 Lecture #27
§3.6: Asymptotic Expansions of Integrals, Part I

Even solutions of simple-looking ODEs can lead to integrals that cannot be evaluated in closed form.

The solution of the second-order linear IVP,

$$y'' + 2\lambda t y' = 0,\ y(0) = 0,\ y'(0) = 1,$$

is

$$y(t;\lambda) = \int_0^t \exp(-\lambda s^2)\ ds.$$

How do we approximate the value of y for fixed λ when t is large, or for fixed t when λ is large?

We introduce standard techniques for approximating certain kinds of integrals.

§3.6.1: Laplace Integrals. One type of integral for which there is approximation technique, is the integral of the form

$$I(\lambda) = \int_a^b f(t)e^{-\lambda g(t)}\ dt,\ \lambda \gg 1$$

where g is strictly increasing function on $[a, b]$ and g' is continuous on $[a, b]$.

We assume that $a < b \leq \infty$.

We may think of $\lambda \gg 1$ as meaning $\lambda \to \infty$.

An example of this type of integral is the Laplace transform

$$U(\lambda) = \int_0^\infty f(t)e^{-\lambda t}\ dt.$$

The type of integral described by $I(\lambda)$ can always be transformed into an integral of the form $U(\lambda)$.

The change of variable $s = g(t) - g(a)$ is invertible because g is increasing, giving the existence of a function $h(s)$ such that $t = h(s)$.

The differentials dt and ds are related by $ds/dt = g'(t)$, or

$$dt = \frac{1}{g'(h(s))}ds$$

where by assumption, g' is strictly positive and continuous.

Applying the change of variable to the integral $I(\lambda)$ gives

$$I(\lambda) = \int_0^{g(b)-g(a)} f(h(s))\exp\left(-\lambda(s + g(a))\right) ds = e^{-\lambda g(a)} \int_0^{g(b)-g(a)} \frac{f(h(s))}{g'(h(s))}e^{-\lambda s}\ ds.$$

The technique, known as Laplace's method, for approximating an integral of the form

$$I(\lambda) = \int_a^b f(t)e^{-\lambda t}\, dt$$

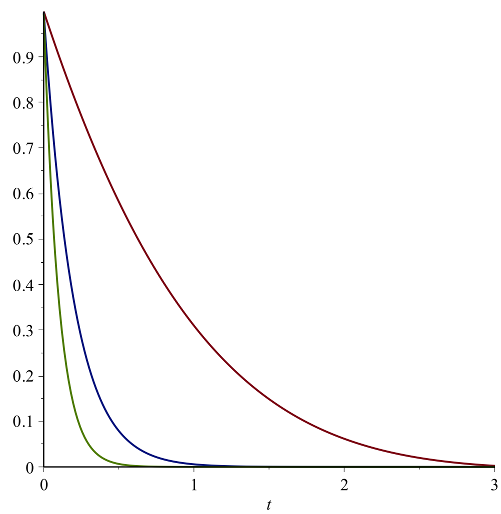
is to identity a subinterval of $[a, b]$ which gives the dominant contribution to the integral.

Because $e^{-\lambda t}$ goes to 0 rapidly as $t \to \infty$, we expect the dominant contribution to come from a subinterval near $t = 0$ as long as $f(t)$ doesn't grow too fast for large t.

Example 3.17. Find an approximation for the integral

$$I(\lambda) = \int_0^\infty \frac{\sin t}{t} \exp(-\lambda t)\, dt.$$

Here are the graphs of the integrand for $\lambda = 1, 2, 10$ (the top or red, the middle or blue, and the bottom or green graph).



From these graphs we see that the dominant contribution comes from an small subinterval containing 0.

For some $T > 0$ we split the integral into

$$I(\lambda) = \int_0^T \frac{\sin t}{t} \exp(-\lambda t)\, dt + \int_T^\infty \frac{\sin t}{t} \exp(-\lambda t)\, dt.$$

For $T > 1$, the second integral is an **exponentially small term** (denoted by EST) because

$$\left| \int_T^\infty \frac{\sin t}{t} \exp(-\lambda)\, dt \right| \leq \int_0^T \left| \frac{\sin t}{t} \right| \exp(-\lambda t)\, dt \leq \int_T^\infty \exp(-\lambda t)\, dt = \frac{\exp(-\lambda T)}{\lambda}.$$

This shows that the second integral is $O(\lambda^{-1} \exp(-\lambda T))$ as $\lambda \to \infty$.

The exponential small term is better yet $o(\lambda^{-m})$ for any $m \in \mathbb{N}$, so that it decays faster to zero than any negative positive of λ.

Thus we write

$$I(\lambda) = \int_0^T \frac{\sin t}{t} \exp(-\lambda t)\ dt + \text{EST}.$$

On the finite interval $[0, T]$ we replace $\sin t$ by the first few terms of its Taylor's series,

$$\sin t = t - \frac{t^3}{3!} + O(t^5),$$

so that

$$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + O(t^4).$$

Then we have that

$$I(\lambda) = \int_0^T \left(1 - \frac{t^2}{3!} + O(t^4)\right) \exp(-\lambda t)\ dt + \text{EST}.$$

Making the change of variable $u = \lambda t$ results in

$$I(\lambda) = \frac{1}{\lambda} \int_0^{\lambda T} \left(1 - \frac{u^2}{3!\lambda^2} + O\left(\frac{1}{\lambda^4}\right)\right) \exp(-u)\ du + \text{EST}.$$

At this point we can let $T \to \infty$ and ignore the EST to get

$$I(\lambda) \approx \frac{1}{\lambda} \int_0^{\infty} \left(1 - \frac{u^2}{3!\lambda^2} + O\left(\frac{1}{\lambda^4}\right)\right) \exp(-u)\ du.$$

Using the integration formulas

$$\int_0^\infty u^m e^{-u}\ du = m!,\ \ m = 0, 1, 2, 3, \ldots,$$

we obtain the asymptotic expansion

$$I(\lambda) \sim \frac{1}{\lambda} - \frac{2!}{3!\lambda^3} + O\left(\frac{1}{\lambda^5}\right),\ \ \lambda \gg 1.$$

We have used here the **gamma function**,

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\ du,\ \ x > 0$$

which satisfies $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(x+1) = x\Gamma(x)$, and $\Gamma(m) = (m-1)!$ for $m = 1, 2, 3, \ldots$.

We state a result and its proof for the asymptotical expansion of a large number of integrals.

Watson's Lemma. If, for the integral

$$I(\lambda) = \int_0^b t^\alpha h(t) e^{-\lambda t}\ dt,$$

we have $\alpha > -1$, $h(t)$ has a Taylor expansion about $t = 0$ with $h(0) \neq 0$, and $|h(t)| < ke^{ct}$ on $0 < t < b$ for some positive constants k and c, then we have the asymptotic expansion

$$I(\lambda) \sim \sum_{n=0}^{\infty} \frac{h^{(n)}(0)\Gamma(\alpha + n + 1)}{n! \lambda^{\alpha+n+1}}, \ \lambda \gg 1.$$

Proof. The condition $\alpha > -1$ guarantees the convergence of the improper integral at $t = 0$.

The exponential boundedness of h guarantees the convergence of the improper integral as $t \to \infty$.

We split the integral at some $T > 0$ to get

$$I(\lambda) = \int_0^T t^\alpha h(t) \exp(-\lambda t) \ dt + \int_T^\infty t^\alpha h(t) \exp(-\lambda t) \ dt.$$

The second integral is an exponential sum term, so that we have

$$I(\lambda) = \int_0^T t^\alpha h(t) \exp(-\lambda t) \ dt + \text{EST}.$$

Replacing $h(t)$ with its Taylor series at $t = 0$ gives

$$\begin{aligned} I(\lambda) &= \int_0^T t^\alpha \left(h(0) + h'(0)t + \frac{h''(0)t^2}{2!} + \cdots \right) \exp(-\lambda t) \ dt + \text{EST} \\ &= \int_0^T \left(h(0)t^\alpha + h'(0)t^{\alpha+1} + \frac{h''(0)t^{\alpha+2}}{2!} + \cdots \right) \exp(-\lambda t) \ dt + \text{EST}. \end{aligned}$$

Making the substitution $u = \lambda t$, ignoring the EST, and replacing the upper limit of integration with ∞ gives the asymptotical expansion

$$I(\lambda) \sim \frac{1}{\lambda} \int_0^\infty \left(h(0) \left(\frac{u}{\lambda}\right)^\alpha + h'(0) \left(\frac{u}{\lambda}\right)^{\alpha+1} + \frac{h''(0)}{2!} \left(\frac{u}{\lambda}\right)^{\alpha+2} + \cdots \right) \exp(-\lambda t) \ dt.$$

Finally, making use of the Gamma function gives the asymptotic expansion. $\square$