Math 521 Lecture #28
§3.6: Asymptotic Expansions of Integrals, Part II

Recall Watson's Lemma that under certain hypotheses, there holds

$$\int_0^b t^\alpha h(t)\exp(-\lambda t)\ dt \sim \sum_{n=0}^\infty \frac{h^{(n)}(0)\Gamma(\alpha+n+1)}{n!\lambda^{\alpha+n+1}},\ \lambda \gg 1.$$

Example 3.19. The complementary error function is

$$\operatorname{erfc}(\lambda) = \frac{2}{\sqrt{\pi}}\int_\lambda^\infty \exp(-s^2)\ ds.$$

To apply Watson's Lemma to this, we first make the change of variable $t = s - \lambda$. The integral becomes

$$\begin{aligned}\operatorname{erfc}(\lambda) &= \frac{2}{\sqrt{\pi}}\int_0^\infty \exp\big(-(t+\lambda)^2\big)\ dt\\ &= \frac{2e^{\lambda^2}}{\sqrt{\pi}}\int_0^\infty \exp\big(-t^2-2\lambda t\big)\ dt\\ &= \frac{2e^{\lambda^2}}{\sqrt{\pi}}\int_0^\infty \exp\big(-t^2\big)\exp\big(-2\lambda t\big)\ dt.\end{aligned}$$

Second, we make the change of variable $\tau = 2t$, so that

$$\operatorname{erfc}(\lambda) = \frac{e^{-\lambda^2}}{\sqrt{\pi}}\int_0^\infty \exp\left(-\frac{\tau^2}{4}\right)\exp(-\lambda\tau)\ d\tau.$$

Since

$$h(\tau) = \exp\left(-\frac{\tau^2}{4}\right) = \sum_{n=0}^\infty \frac{(-1)^n\tau^{2n}}{n!4^n} = 1 - \frac{\tau^2}{4} + \frac{\tau^4}{2!4^2} - \frac{\tau^6}{3!4^3} + \cdots,$$

we have $h^{(n)}(0) = 0$ for n odd and $h(0) = 1$, $h^{(2)}(0) = -1/2$, $h^{(4)}(0) = 3/2^2$, $h^{(6)} = -15/2^3$, etc.

By Watson's Lemma with $\alpha = 0$, we obtain

$$\begin{aligned}\operatorname{erfc}(\lambda) &\sim \frac{e^{-\lambda^2}}{\sqrt{\pi}}\left(\frac{1}{\lambda} - \frac{\Gamma(3)}{2!2\lambda^3} + \frac{3\Gamma(5)}{4!2^2\lambda^5} - \frac{15\Gamma(7)}{6!2^3\lambda^7} + \cdots\right)\\ &= \frac{e^{-\lambda^2}}{\sqrt{\pi}}\left(\frac{1}{\lambda} - \frac{\Gamma(3)}{2^2\lambda^3} + \frac{\Gamma(5)}{2!2^4\lambda^5} - \frac{\Gamma(7)}{3!2^6\lambda^7} + \cdots\right)\\ &= \frac{2e^{-\lambda^2}}{\sqrt{\pi}}\left(\frac{1}{2\lambda} - \frac{\Gamma(3)}{(2\lambda)^3} + \frac{\Gamma(5)}{2!(2\lambda)^5} - \frac{\Gamma(7)}{3!(2\lambda)^7} + \cdots\right)\\ &= \frac{2e^{-\lambda^2}}{\sqrt{\pi}}\left(\frac{1}{2\lambda} - \frac{1}{4\lambda^3} + \frac{3}{8\lambda^5} - \frac{15}{16\lambda^7} + \cdots\right)\end{aligned}$$

when $\lambda \gg 1$.

§3.6.2: Integration by Parts. Sometimes an asymptotic expansion can be found by successive uses of integration by parts.

A drawback of this method is that is requires more work and is not as applicable as the Laplace method.

We can apply successive integration by parts to $\operatorname{erfc}(\lambda)$, in which the first step gives

$$\begin{aligned}
\operatorname{erfc}(\lambda) &= \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} \exp(-t^2)\ dt \\
&= \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} \frac{-2t\exp(-t^2)}{-2t} dt,\ [u = \frac{1}{-2t},\ dv = -2t\exp(-t^2)dt], \\
&= \frac{2}{\sqrt{\pi}} \left[\frac{\exp(-t^2)}{-2t} - \int \frac{\exp(-t^2)}{2t^2} dt \right]_{\lambda}^{\infty} \\
&= \frac{2}{\sqrt{\pi}} \left[\frac{\exp(-\lambda^2)}{2\lambda} - \int_{\lambda}^{\infty} \frac{\exp(-t^2)}{2t^2} dt \right] \\
&= \frac{2e^{-\lambda^2}}{\sqrt{\pi}} \left(\frac{1}{2\lambda} \right) - \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} \frac{\exp(-t^2)}{2t^2} dt.
\end{aligned}$$

We have obtained the first term we got by the Laplace method.

Applying integration by parts to the new integral gives

$$\begin{aligned}
\int_{\lambda}^{\infty} \frac{\exp(-t^2)}{2t^2} dt &= \frac{1}{2} \int_{\lambda}^{\infty} \frac{-2t\exp(-t^2)}{-2t^3} dt\ [u = -\frac{1}{2t^3},\ dv = -2t\exp(-t^2)dt] \\
&= \frac{1}{2} \left[\frac{\exp(-t^2)}{-2t^3} - \int \frac{3\exp(-t^2)}{2t^4} dt \right]_{\lambda}^{\infty} \\
&= \frac{1}{2} \left[\frac{\exp(-\lambda^2)}{2\lambda^3} - \int_{\lambda}^{\infty} \frac{3\exp(-t^2)}{2t^4} dt \right].
\end{aligned}$$

Thus we have

$$\operatorname{erfc}(\lambda) = \frac{e^{-\lambda^2}}{\sqrt{\pi}} \left(\frac{1}{2\lambda} - \frac{1}{4\lambda^3} - \int_{\lambda}^{\infty} \frac{3\exp(-t^2)}{2t^4} dt \right),$$

obtaining the second term we got by the Laplace method.

Continuing with successive integration by parts we will obtain the asymptotic expansion we did by the Laplace method.

With any method to find an asymptotical expansion, it is important to ensure that each succeeding term is the expansion is asymptotically smaller than the preceding terms.

The asymptotical expansion we have obtained for $\operatorname{erfc}(\lambda)$ with the terms

$$a_n = \frac{(-1)^n \Gamma(2n+1)}{n!(2\lambda)^{2n+1}} = \frac{(-1)^n (2n)!}{n!(2\lambda)^{2n+1}},\ n \geq 1$$

is, for a fixed value of λ, a divergent series because, by the ratio test, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(2\lambda)^2} = \infty.$$

How do we use the divergent series to obtain good approximations?

Rather than taking more and more terms, we fix the number n of terms we use from the divergent series, and apply the approximation to large values of λ.

The error in this approximation is about the value of the $(n+1)^{\text{th}}$-term.

For example, we have

$$\operatorname{erfc}(2) = \frac{2e^{-4}}{\sqrt{\pi}}\left(\frac{1}{2^2} - \frac{1}{2^5} + \frac{3}{2^8} - \frac{15}{2^{11}} + \frac{105}{2^{14}} - \frac{945}{2^{17}} + \cdots\right).$$

After the fifth term, the magnitude of the terms begins to increase.

Adding up the first five terms gives the $\operatorname{erfc}(2) \approx 0.004744$ which when compared with a more precise approximation of 0.004678 has only an error of roughly 0.000066.

Instead of the asymptotic expansion we could written

$$\operatorname{erfc}(\lambda) = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp(-t^2)\,dt - \frac{2}{\sqrt{\pi}}\int_0^\lambda \exp(-t^2)\,dt = 1 - \frac{2}{\sqrt{\pi}}\int_0^\lambda \exp(-t^2)\,dt,$$

and used the Taylor series for $\exp(-t^2)$ to get

$$\begin{aligned}\operatorname{erfc}(\lambda) &= 1 - \frac{2}{\sqrt{\pi}}\int_0^\lambda \left(1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots\right) dt \\ &= 1 - \frac{2}{\sqrt{\pi}}\left(\lambda - \frac{\lambda^3}{3} + \frac{\lambda^5}{10} - \frac{\lambda^7}{42} + \cdots\right).\end{aligned}$$

It takes 20 terms of this to get that same degree of accuracy as the divergent asymptotic expansion did with 5 terms.

Asymptotic expansions, although divergent, can be significantly more efficient than Taylor series.

§3.6.3: Other Integrals. We can extend these asymptotic ideas to approximate other integrals of the form

$$I(\lambda) = \int_a^b f(t)\exp\big(\lambda g(t)\big)\,dt,\ \lambda \gg 1,$$

where f is continuous and g is sufficiently smooth with a unique maximum at $t = c$ in (a, b).

Then $g'(c) = 0$ and $g''(c) < 0$.

The main contribution to $I(\lambda)$ comes from an interval in which g attains it maximum value.

To obtain a leading-order approximation, we replace $f(t)$ by $f(c)$ and $g(t)$ by its second-order Taylor polynomial about $t = c$ to get

$$I(\lambda) \approx \int_a^b f(c)\exp\left(\lambda g(c) + \frac{\lambda g''(c)(t-c)^2}{2}\right) dt.$$

This simplifies to

$$I(\lambda) \approx f(c)\exp(\lambda g(c)) \int_a^b \exp\left(\frac{\lambda g''(c)(t-c)^2}{2}\right) dt.$$

The change of variable

$$s = (t-c)\sqrt{\frac{-\lambda g''(c)}{2}},\ ds = \sqrt{\frac{-\lambda g''(c)}{2}} dt,$$

results in

$$I(\lambda) \approx f(c)\exp(\lambda g(c))\sqrt{\frac{2}{-\lambda g''(c)}} \int_{(a-c)\sqrt{-\lambda g''(c)/2}}^{(b-c)\sqrt{-\lambda g''(c)/2}} \exp(-s^2)\ ds.$$

Replacing the interval of integration by $(-\infty, \infty)$ and using

$$\int_{-\infty}^{\infty} \exp(-s^2)\ ds = \sqrt{\pi}$$

gives

$$I(\lambda) \approx f(c)\exp(\lambda g(c))\sqrt{\frac{2\pi}{-\lambda g''(c)}},\ \lambda \gg 1.$$

If, instead, the maximum value of g occurs at either endpoint of (a, b), then one of the limits of integration will be 0 in the integral in s, resulting in one-half of the approximating value.