Math 521 Lecture #29
§4.1: Variational Problems

The calculus of variations deals with the optimization of variable quantities, called *functionals*, over some admissible class of competing objects.

Many of the techniques in the calculus of variations were developed by Euler and Lagrange.

These techniques provide many important methods that are used in applied mathematics.

§4.1.1: Functionals.
The simplest optimization problem deals with a real-valued function f of a single real variable belonging to a prescribed interval I.

We will focus on the optimization problem of minimization. [The maximization problem is similar.]

Recall that f has a local minimum at $x_0 \in I$ if there exists $\delta > 0$ such that $f(x_0) \leq f(x)$ for all $x \in I$ satisfying $|x - x_0| < \delta$.

If x_0 is an interior point of I, and if f is differentiable at x_0, then $f'(x_0) = 0$ and a **critical or stationary point** for f is x_0.

Having a critical point is a **necessary condition** for a local minimum, but is not sufficient (as for $f(x) = x^3$ with critical point $x_0 = 0$).

If f is twice differentiable at x_0, then a sufficient condition for f to have a local minimum at x_0 is that $f''(x_0) > 0$.

Calculus of variations of a generalization of this problem, where instead of real numbers x the inputs are functions $y(x)$ belonging to a collection A of functions, and the "function" or **functional** f associates to each $y(x) \in A$ a real number $f(y)$.

For the functional, we will typically use a letter like J, instead of f.

The collection A of functions is called a **set of admissible functions**.

The properties of the functions $y(x)$ required to evaluate the functional $J(y)$ determine a set of admissible functions A.

For example, to evaluate the functional

$$J(y) = \int_0^1 \left[\frac{(y'(x))^2}{2} + V(y(x)) \right] dx,$$

we require the function $y(x)$ to be continuously differentiable on $[0, 1]$.

The space of admissible functions for the functional J is

$$A = \{y(x) \in C^1[0, 1]\}.$$

We may impose other restrictions on the functions in A such as $y(0) = 0$.

Which function y^*, if any, in A gives a minimum value for $J(y)$?

We can think of minimum as a local minimum if we have some way to measure the distance $d(y_1, y_2)$ between to functions in A, so that $J(y_0)$ is a local minimum if there exists $\delta > 0$ such that $J(y^*) \leq J(y)$ for all $y \in A$ such that $d(y, y^*) < \delta$.

Or we can think of minimum in the global sense in that

$$J(y^*) \leq J(y) \text{ for all } y \in A.$$

The problem of extremizing a functional J over an admissible set A is called a **variational problem**.

Typically, the functionals are defined by integration, and the main objects of the calculus of variations is to find necessary and sufficient conditions for the existence for extrema.

We will focus on necessary conditions for the existence of extrema. [Finding sufficient conditions for the existence of extrema is a much harder problem and involves specialized function spaces.]

§4.1.2: Examples. We look at two examples that illustrate classical variational problems, their functionals, and their set of admissible functions, where the type of functional $J(y)$ is defined by integration of a **Lagrangian** $L(x, y(x), y'(x))$:

$$J(y) = \int_a^b L(x, y(x), y'(x))\, dx$$

where L is twice continuously differentiable in each of its three arguments.

Example 4.1. Let A be the set of continuously differentiable functions $y(x)$ defined on $[a, b]$ (with $a < b$) that satisfy the conditions $y(a) = y_0$ and $y(b) = y_1$.

Notationally we write

$$A = \{y(x) \in C^1[a, b] : y(a) = y_0, y(b) = y_1\}.$$

Define a functional on A by

$$J(y) = \int_a^b \sqrt{1 + [y'(x)]^2}\, dx.$$

What is $J(y)$ measuring? The arc length of the continuously differentiable curve $y(x)$ connecting the points (a, y_0) and (b, y_1).

The variational problem associated with J is to minimize J (the arc length) over the set A, i.e. to find $y^* \in A$ for which $J(y^*)$ is a global minimum.

We have here what is called a **fixed endpoint** problem: the endpoints of the curves in A are fixed.

We know that the continuously differentiable curve y^* of shortest arc length is the straight line connecting the two points:

$$y^*(x) = \frac{y_1 - y_0}{b - a}(x - a) + y_0,\ \ a \leq x \leq b.$$

Example 4.2. A bead of mass m with initial velocity zero slides with no friction under the force due to gravity g from a point $(0, b)$ to a point $(a, 0)$ along a wire defined by a curve $y(x)$.

Which curve $y(x)$ leads to the shortest time of descent? [This was proposed by Bernoulli in 1696, and solved later by Euler.]

Here is how we formulate this variational problem.

For a fixed curve $y(x)$ satisfying $y(0) = b$ and $y(a) = 0$, let $s(x)$ denote the arc length along the curve starting from $(0, b)$, so that $S = s(a)$ is the total arc length of the curve $y(x)$.

Presumably, there is a continuously differentiable relationship between s and the time t it takes the bead to reach the point on the curve a distance s from the start: $t = g(s)$.

The time of descent T along $y(x)$ thus satisfies

$$T = \int_0^T dt = \int_0^S g'(s)\ ds = \int_0^S \frac{dt}{ds}\ ds.$$

Since ds/dt is the velocity v of the bead, we have that

$$T = \int_0^S \frac{ds}{v}.$$

Since s is the arc length along the curve $y(x)$, we have $ds = \sqrt{1 + [y'(x)]^2}\ dx$, and obtain

$$T = \int_0^a \frac{\sqrt{1 + [y'(x)]^2}}{v}\ dx.$$

It remains to express v as function of x which we do through the no friction assumption and the only force acting on the bead being due to gravity: there is a conserved total energy given by

$$E = \frac{mv^2}{2} + mgy.$$

We know what v and y are when $x = 0$, namely $v(0) = 0$ and $y(0) = b$, so that

$$\frac{mv^2}{2} + mgy = mgb.$$

Solving for the velocity gives

$$v = \sqrt{2g(b - y(x))}.$$

We arrive at the functional

$$T(y) = \int_0^a \sqrt{\frac{1 + [y'(x)]^2}{2g(b - y(x))}}\ dx$$

The set of admissible curves for this functional is

$$A = \left\{ y(x) \in C^1[0,a] : y(0) = b, y(a) = 0, y(x) \leq b, \int_0^a \frac{dx}{\sqrt{b - y(x)}} < \infty \right\}.$$

The last condition is needed to have the improper integral for $T(y)$ be convergent.

Is there a global minimizer $y^*(x) \in A$ for $T(y)$?

Solving this is a nontrivial problem, and the answer is that $y^*(x)$ is an arc of a cycloid, the curve traced out by a point on the rim of a moving circle, also known as the **brachistochrone**.